Looking for Pythagoras

The Pythagorean Theorem

Goals of the Unit

- Relate the area of a square to the side length
- Estimate the values of square roots of whole numbers
- Locate irrational numbers on a number line
- Develop strategies for finding the distance between two points on a coordinate grid
- Understand and apply the Pythagorean Theorem
- Use the Pythagorean Theorem to solve everyday problems

Developing Students' Mathematical Habits

The overall goal of *Connected Mathematics* is to help students develop sound mathematical habits. Through their work in this and other geometry units, students learn important questions to ask themselves about any situation that can be represented and modeled mathematically, such as

- *Is it appropriate and useful to use the Pythagorean Theorem in this situation? How do I know this?*
- *Do I need to find the distance between two points?*
- *What are the quantities in this problem?*
- *How can I estimate the square root of a number?*
- *How can I find the length of something without directly measuring it?*

Mathematics of the Unit

Pearson Prentice Hall Professional Development

Overview

In *Looking for Pythagoras*, students explore two important ideas: the Pythagorean Theorem and square roots. They also review and make connections among the concepts of area, distance, and irrational numbers.

Students begin the unit by finding the distance between points on a coordinate grid. They learn that the positive square root of a number is the side length of a square whose area is that number. Then, students discover the Pythagorean relationship through an exploration of squares drawn on the sides of a right triangle. In the last investigation of the unit, students apply the Pythagorean Theorem to a variety of problems.

Summary of Investigations

Investigation 1

Coordinate Grids

Students review coordinate grids as they analyze a map in which streets are laid out on a grid. They make the connection between the coordinates of two points and the driving distance between them. This sets the stage for finding the distance between two points on a grid without measuring. Students investigate geometric figures on coordinate grids. Given two vertices, they find other vertices that define a square, a non-square rectangle, a right triangle, and a non-rectangular parallelogram. And, they calculate areas of several figures drawn on a dot grid.

Investigation 2

Squaring Off

Students explore the relationship between the area of a square drawn on a dot grid and the length of its sides. This provides an introduction to the concept of square root. They find the distance between two points by analyzing the line segment between them: they draw a square using the segment as one side, find the area of the square, and then find the positive square root of that area.

Investigation 3

The Pythagorean Theorem

Students develop and explore the Pythagorean Theorem. They then investigate a geometric puzzle that verifies the theorem, and they use the theorem to find the distance between two points on a grid. In the last problem, they explore and apply the converse of the Pythagorean Theorem.

Investigation 4

Using the Pythagorean Theorem

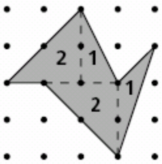
For students to appreciate the mathematical power of the Pythagorean Theorem, they need to encounter situations that can be illuminated by the theorem. Students explore an interesting pattern among right triangles, apply the Pythagorean Theorem to find distances on a baseball diamond, investigate properties of 30-60-90 triangles, and find missing lengths and angle measures of a triangle composed of smaller triangles.

Mathematics Background

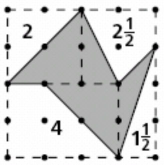
Students' work in this unit develops an important relationship connecting geometry and algebra: the Pythagorean Theorem. The presentation of ideas reflects the historical development of the concept of irrational numbers. Early Greek mathematicians searched for ratios of integers to represent side lengths of squares with certain given areas such as 2 square units. The square root of 2 is an irrational number, which means that it cannot be written as a ratio of two integers.

Finding Area and Distance

Students find areas of plane figures drawn on dot grids. This reviews some concepts developed in the grade 6 unit *Covering and Surrounding*. One common method for calculating the area of a figure is to subdivide it and add the areas of the component shapes. A second common method is to enclose the shape in a rectangle and subtract the areas of the shapes that lie outside the figure from the area of the rectangle. Below, the area of the shape is found with each method.

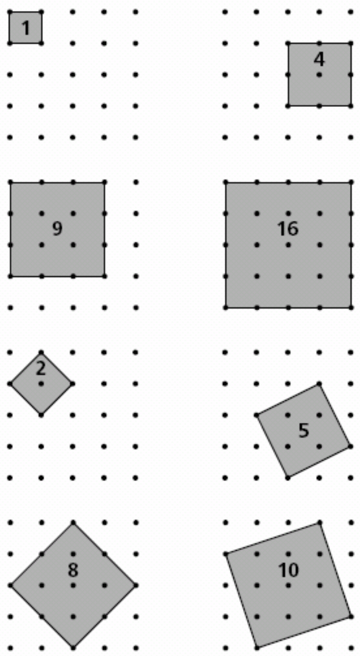


Subdivide to find the area:
$2 + 2 + 1 + 1 = 6$



Enclose in a square to find the area:
$16 - (4 + 2 + 2\frac{1}{2} + 1\frac{1}{2}) = 6$

In Investigation 2, students draw squares with as many different areas as possible on a 5 dot-by-5 dot grid. There are eight possible squares, four "upright" and four "tilted."

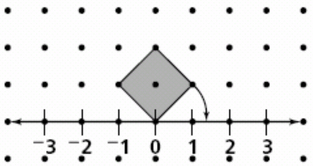


Square Roots

If the area of a square is known, its side length is easy to determine: it is the number whose square is the area. The fact that some of these lengths are not whole numbers prompts the introduction of the $\sqrt{\ }$ symbol. The lengths of the sides of the preceding squares (in units) are 1, 2, 3, 4, $\sqrt{2}$, $\sqrt{5}$, $\sqrt{8}$, and $\sqrt{10}$. Because the grid is a centimeter grid, students can estimate the values of the square roots by measuring these lengths with a ruler. By making these ruler estimates and comparing them to estimates obtained by computing square roots on a calculator, students develop a sense of these numbers and begin to realize that they cannot be expressed as terminating or repeating decimals.

Students also develop benchmarks for estimating square roots. For example, $\sqrt{5}$ is between 2 and 3 because $4 < 5 < 9$, and since 5 is closer to 4 than 9, we estimate that $\sqrt{5}$ is closer to 2 than 3. Students might try 2.25. But $2.25^2 = 5.06$. So, $\sqrt{5}$ is between 2 and 2.25, but closer to 2.25. They might try 2.24 to get $2.24^2 = 5.0176$, which is closer. This method can be continued until the desired accuracy is obtained. Students also estimate square roots with a number line ruler, which helps them to develop a sense of the size of the irrational numbers such as $\sqrt{3}$, $\sqrt{5}$, and $\sqrt{7}$. One way to locate $\sqrt{2}$ on the number line is as follows:

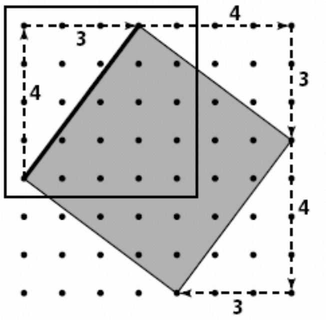
The square below has an area of 2 square units. The length of a side of this square is $\sqrt{2}$ units. If we draw a number line as shown, and use a compass to mark off a segment with the same length as a side of the square, we can see that the segment is about 1.4 units long.



Using Squares to Find Lengths of Segments

Finding the areas of squares leads students to a method for finding the distance between two dots. The distance between two dots on a dot grid is the length of the line segment connecting them. To find this length, students can draw a square with the segment as one side. The distance between the two dots is the square root of the area of the square.

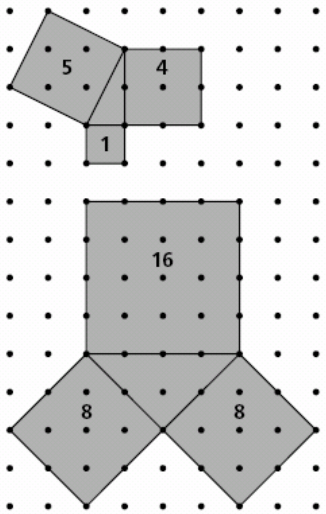
To use this method to find all the different lengths of line segments that can be drawn on a 5 dot-by-5 dot grid, the grid must be extended to fit the squares associated with those lengths. For example, the bold line segment below is the side of a square (shaded) with an area of 25 square units, so the segment has length $\sqrt{25}$ units, or 5 units.



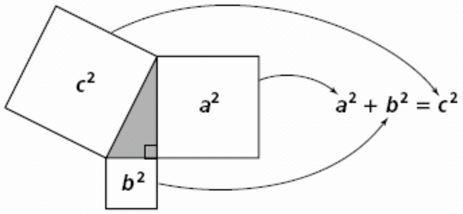
To draw the square with the given side length, many students will use an "up and over" or "down and over" method to go from one point to the next. For example, to get from the lower endpoint of the segment above to the other endpoint, you go up 4 units and right 3 units. These endpoints are two vertices of the square. To get the third vertex, go right 4 units and down 3. To get the fourth, go down 4 units and to the left 3. In this way, they are developing intuition about the Pythagorean Theorem.

Developing and Using the Pythagorean Theorem

Once students are comfortable with finding the length of a segment by thinking of it as the side of a square, they investigate the patterns among the areas of the three squares that can be drawn on the sides of a right triangle.



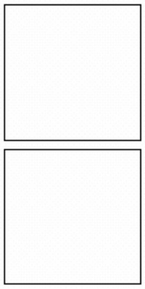
The observation that the square on the hypotenuse has an area equal to the sum of the areas of the squares on the legs leads students to the Pythagorean Theorem: If a and b are the lengths of the legs of a right triangle and c is the length of the hypotenuse, then $a^2 + b^2 = c^2$.



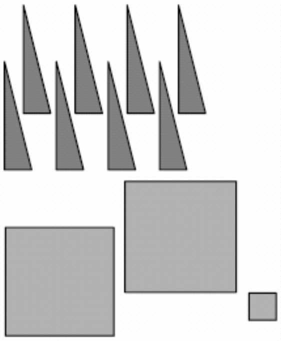
A *theorem* is a general mathematical statement that has been proven true. Over 300 different proofs have been given for the Pythagorean Theorem. It is regarded as one of the most important developments in mathematics because it allows us to link ideas of number to ideas of space.

A Proof of the Pythagorean Theorem

Students solve a puzzle that gives a geometric proof of the Pythagorean Theorem. The puzzle pieces consist of eight congruent right triangles and three squares.

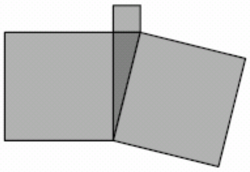


Puzzle frames



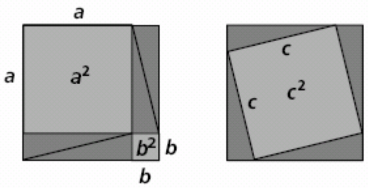
Puzzle pieces

The side lengths of the squares are the lengths of the three triangle sides.



To solve the puzzle, students must arrange the pieces to fit into two square puzzle frames. Students' arrangements of the 11 shapes may differ slightly, but all arrangements lead to the same conclusion.

One possible arrangement is shown below. The sides' lengths of the right triangle have been labeled a, b, and c.



Once the shapes are arranged, you can reason as follows:

- The areas of the frames are equal. They are squares with side lengths of $a + b$.
- Each frame contains four identical right triangles. The other shapes are squares with area a^2, b^2, and c^2.
- If the four right triangles are removed from each frame, the area remaining in the two frames must be equal. That is, the sum of the areas of the squares in one frame must equal the area of the square in the other frame.

Geometrically, the diagram shows that if the lengths of the legs of a right triangle are a and b, and the length of the hypotenuse is c, then $a^2 + b^2 = c^2$. You can make similar puzzle pieces starting with any right triangle and then arrange the shapes in the same way. Therefore, this statement is true for any right triangle.

In later courses, students may see this geometric argument presented algebraically. The sum of the areas of the two squares and the four triangles in the left frame equals the sum of the areas of the square and the four triangles in the right frame:

$$a^2 + b^2 + 4\left(\frac{ab}{2}\right) = c^2 + 4\left(\frac{ab}{2}\right)$$
$$a^2 + b^2 = c^2$$

The Pythagorean Theorem has many applications that connect the concepts of line segment lengths, squares, and right angles.

Using the Pythagorean Theorem to Find Lengths

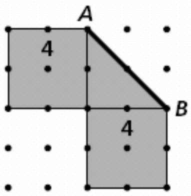
Students use the Pythagorean Theorem to find the distance between two dots on a dot grid. The length of a horizontal or vertical line segment drawn on a dot grid can be found by counting the units directly. If the segment is not vertical or horizontal, it is always possible to treat it as the hypotenuse of a right triangle with vertical and horizontal legs. The length of the hypotenuse—and thus the distance between the dots—can then be found with the Pythagorean Theorem.

In high school, students will see the following formula for finding the distance between two points, (x_1, y_1) and (x_2, y_2) in the plane:

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

This is simply the Pythagorean Theorem where $a = x_1 - x_2$ (the horizontal distance between two points), $b = y_1 - y_2$ (the vertical distance between two points), and $c = d$.

To find the length of line segment AB below, draw a right triangle with segment AB as the hypotenuse. Calculate the areas of the squares on the legs of the triangle (4 square units each), add these areas (8 square units, which is the area of the square drawn on the hypotenuse), and take the square root. The length of AB is $\sqrt{8}$ units.



The Converse of the Pythagorean Theorem

The *converse* of a statement of the form "If p then q" is "If q then p." The converse of the Pythagorean Theorem states: If a, b, and c are the lengths of the sides of a triangle and $a^2 + b^2 = c^2$, then the triangle is a right triangle. The converse of a true statement is not always true. However, the converse of the Pythagorean Theorem is true and can be used to show that a given triangle is a right triangle. For example, if you know the side lengths of a triangle are 6 in., 8 in., and 10 in., then because $6^2 + 8^2 = 10^2$, you can conclude that the triangle is a right triangle.

Students do not formally prove the converse of the Pythagorean Theorem in this unit. Rather, they build triangles with a variety of different side lengths and determine whether they are right triangles. Based on their findings, they conjecture that triangles whose side lengths satisfy $a^2 + b^2 = c^2$ are right triangles.

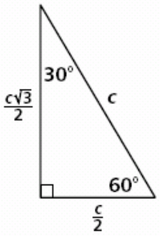
Students are asked to explain why their conjecture is true. One explanation is: "Suppose we know that Triangle 1 has sides a, b, and c, that satisfy the relationship $a^2 + b^2 = c^2$. Suppose Triangle 2 has sides a, b, and d and we know that Triangle 2 is a right triangle with leg lengths of a and b. Then $a^2 + b^2 = d^2$. From the first statement we know that $a^2 + b^2 = c^2$. Logically, this gives us that $c^2 = d^2$, and, therefore, $c = d$ (because they must both be positive numbers). Now Triangle 1 and Triangle 2 have the same three measures for their sides. In *Shapes and Designs,* students learned that once you know all three sides of a triangle, it is uniquely identified. They will investigate this idea more formally when they study congruence of triangles in *Hubcaps, Kaleidoscopes, and Mirrors.* So these two triangles are identical, right-angled triangles. In other words it is impossible for a triangle whose sides fit the relationship $a^2 + b^2 = c^2$ to not be a right-angled triangle.

An interesting byproduct of the converse of the Pythagorean Theorem is the concept of *Pythagorean triples,* sets of numbers that satisfy the relationship $a^2 + b^2 = c^2$. Students discover that finding Pythagorean triples means finding two square numbers whose sum is also a square number. Multiples of one triple will generate countless others. For example, once you establish that 3-4-5 is a Pythagorean triple, you know that 6-8-10, 9-12-15, and so on, are also Pythagorean triples.

Special Right Triangles

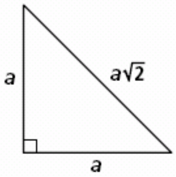
In Investigation 4, students learn about 30-60-90 triangles by starting with an equilateral triangle (a 60-60-60 triangle). They use the line of symmetry to show the reflection line forms two congruent 30-60-90 triangles. For each of these triangles, they deduce that the leg opposite the 30° angle is half the length of the side of the original triangle. They then use the Pythagorean Theorem to find the length of the other leg.

The Pythagorean Theorem can be used to show some special relationships among side lengths of 30-60-90 triangles (that is, triangles with 30°, 60°, and 90° angles).



Suppose the hypotenuse of a 30-60-90 triangle has length c. The length of the side opposite the 30° angle must be half this length, or $\frac{c}{2}$. Using the Pythagorean Theorem, the square of the length of the longer leg is $c^2 - \frac{c^2}{4}$, or $\frac{3c^2}{4}$. So, its length is $\sqrt{\frac{3c^2}{4}}$, or $\frac{c\sqrt{3}}{2}$.

Students also explore isosceles right triangles (45-45-90 triangles), and find that the length of the hypotenuse is always the length of one of the legs times $\sqrt{2}$. If the length of each leg is a then, by the Pythagorean Theorem, the square of the length of the hypotenuse must be $a^2 + a^2$, or $2a^2$. Therefore, the length of the hypotenuse is $\sqrt{2a^2} = a\sqrt{2}$.



Rational and Irrational Numbers

When we examine patterns in the decimal representations of fractions, or rational numbers, we find that the decimals either terminate or repeat. For example, $\frac{1}{5}$ is equal to 0.2 (a terminating decimal) and $\frac{1}{3}$ is equal to 0.33333. . . (a repeating decimal). Numbers such as $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{5}$ cannot be expressed as repeating or terminating decimals. Students create line segments with these lengths. For example, $\sqrt{2}$ is the length of the hypotenuse of a right triangle whose legs have length 1. They then locate the lengths on a number line. This procedure helps students to estimate the size of these irrational numbers.

Converting Repeating Decimals to Fractions

Because all repeating decimals are rational numbers, they can be represented as fractions. It is not always obvious, though, what fraction is equivalent to a given repeating decimal. One method for converting a repeating decimal to a fraction involves solving an equation. To convert 12.312312. . . to a fraction, for example, call the unknown fraction N. Thus, $N = 12.312312\ldots$. Multiply both sides of the equation by 1,000 (the power of 10 that moves a complete repeating group to the left of the decimal point), which gives $1{,}000N = 12{,}312.312312\ldots$. Then, subtract the first equation from the second, which gives $999N = 12{,}300$. Therefore, $N = \frac{12{,}300}{999}$, or $12\frac{312}{999}$.

The decimal equivalents of fractions with denominators of 9, 99, 999, and so on, display interesting patterns that can be used to write repeating decimals as fractions. For example, all decimals with a repeating part of one digit, such as 0.111 . . . and 0.222 . . . , can be written as a fraction with 9 in the denominator and the repeated digit in the numerator, such as $\frac{1}{9}$ and $\frac{2}{9}$. Decimals with a repeating part of two digits, such as 0.010101 . . . and 0.121212 . . . , can be written as a fraction with 99 in the denominator and the repeated digits in the numerator, such as $\frac{1}{99}$ and $\frac{12}{99}$.

Proof that $\sqrt{2}$ Is Irrational

In high school, students may prove that $\sqrt{2}$ is not a rational number. Its irrationality can be proved in an interesting way—a proof by contradiction. The proof is given here for the teacher's information.

Assume $\sqrt{2}$ is rational. Then, there exist positive integers p and q such that $\sqrt{2} = \frac{p}{q}$. So, $\sqrt{2}q = p$. Squaring both sides gives $2q^2 = p^2$. From the *Prime Time* unit students learned that all square numbers have an odd number of factors. The reason is that factors of a number come in pairs. In a square number the factors in one of the pairs must be equal, which makes the number of factors for a square number odd. This means that if p and q are positive integers, then p^2 and q^2 each have an odd number of factors. Since $p^2 = 2q^2$, p^2 has the same number of factors as $2q^2$. But $2q^2$ has an even number of factors (The factor 2 plus the odd number of factors of q^2.) This is a contradiction. Therefore, p and q cannot exist with these properties and $\sqrt{2}$ must be irrational.

Square Root Versus Decimal Approximation

Problems involving the Pythagorean Theorem often result in square roots that are irrational numbers. Students at this level are often reluctant to leave numbers in a square root form. For example, rather than give an exact answer of $\sqrt{3}$, they give a decimal approximation, such as 1.732. Some students are not comfortable thinking about square roots as numbers. Although it is important to know the approximate size of an answer, especially in a practical problem, it is sometimes better to give an exact answer, and this often means using square root form. For example, in the study of 30-60-90 triangles,

$$\frac{\text{length of leg opposite the } 60^\circ \text{ angle}}{\text{length of the hypotenuse}} = \frac{\sqrt{3}}{2}$$

Here, $\sqrt{3}$ is much easier to remember than a multi-digit decimal approximation, and the expression using the square root gives the exact result. Similarly, in a right triangle, if the hypotenuse has a length of 9 units and one leg has a length of 8 units, then the length of the other leg is $\sqrt{81 - 64} = \sqrt{17}$ units. This answer is exact, while the calculator answer, 4.123105626, is an approximation. This is not to say that all answers should be left in square root form—context needs to be considered. Heights of buildings are more easily comprehended in whole-number or decimal form, even if that form does not give the precise answer. Students should be encouraged to leave an answer in square root form when there is no practical reason to express it as a decimal approximation. The hope is that all students will become comfortable with square roots as numbers in contexts where expressing an answer as a square root is appropriate. In this unit, we want students to have a "sense" of square roots as numbers and some idea of where they fit on the number line or between what two rational numbers they occur.

Number Systems

New number systems are created when a problem arises that cannot be answered within the system currently in use, or when inconsistencies arise that can be taken care of only by expanding the domain of numbers in the system.

The historical "discoveries" of new number systems in response to needs are reflected in the number sets students use in grades K–12. Elementary students begin with the *counting numbers*, also called *natural numbers*. Then, zero is added to the system to create the set of *whole numbers*. Later, students learn that negative numbers are needed to give meaning in certain contexts, such as temperature. Now they have the number system called the *integers*.

In elementary and middle school, students learn about fractions and situations in which fractions are useful, as in many division problems. Students' number world has been expanded to the set of *rational numbers*.

In this unit, students encounter contexts in which the need for *irrational numbers* arises. Specifically, they need irrational numbers to express the exact lengths of tilted segments on a grid. The set of rational numbers and the set of irrational numbers compose the set of *real numbers*. The diagram in Figure 1 is one way to represent these sets of numbers.

Figure 1