Unit Introduction

Growing, Growing, Growing

Exponential Relationships

Goals of the Unit

- Recognize situations in which one variable is an exponential function of another variable
- Recognize the connections between exponential equations and growth patterns in tables and graphs of those equations
- Construct equations to express exponential patterns that appear in data tables, graphs, and problem conditions
- Understand and apply the rules for operating on numerical expressions with exponents
- Solve problems about exponential growth and decay from a variety of different subject areas, including science and business
- Compare exponential and linear relationships

Developing Students' Mathematical Habits

The overall goal of *Connected Mathematics* is to help students develop sound mathematical habits. Through their work in this and other algebra units, students learn important questions to ask themselves about any situation that can be represented and modeled mathematically, such as:

- *What are the variables?*
- *Is the relationship between variables an example of exponential growth or decay?*
- *How can the relationship be detected in a table, graph, or equation? What is the growth or decay factor?*
- *What equation models the data in the table?*
- *What equation models the pattern in the graph?*
- *What can I learn about this situation by studying a table, graph, or equation of the exponential relationship?*
- *How does the relationship compare to other types of relationships that I have studied?*

Mathematics of the Unit

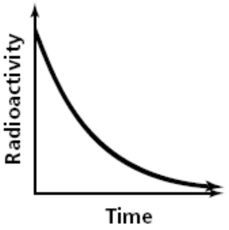
Pearson Prentice Hall Professional Development

Overview

One of the central goals of algebra is describing and reasoning about relationships among quantitative variables. That goal has been addressed in many *Connected Mathematics* units. In grade 6, students looked for connections among sides and angles in regular polygons in *Shapes and Designs* and connections among dimensions, perimeters, and areas of various figures in *Covering and Surrounding*. In grade 7, students explored general techniques for representing quantitative relationships with words, pictures, tables, graphs, and symbols in *Variables and Patterns,* and then, in *Moving Straight Ahead,* focused on an important family of quantitative relationships: those with linear graphs illustrating constant, additive rates of change. In the grade 8 unit *Thinking With Mathematical Models*, students continued to explore linear patterns of change and contrasted these patterns with inverse patterns of change.

This algebra unit, *Growing, Growing, Growing,* focuses students' attention on a family of useful nonlinear relationships: those that model exponential growth and exponential decay. Studies of biological populations, from bacteria and amoebas to mammals (including humans), often reveal exponential patterns of growth. The populations increase over time and at increasing rates of growth. Graphs of the (*time, population*) data curve upward. This same pattern of growth at increasing rates is seen when money is invested in accounts paying compound interest or when inflation is tracked.

In other situations, quantities decline as time passes, but the actual amount of decline diminishes over time (unlike the constant rate of decline for decreasing linear relationships). For example, radioactive substances and many medicines decay in nonlinear patterns in the body. These patterns of change are multiplicative. As the independent variable changes by a constant amount, the dependent variable changes by a constant factor.



Summary of Investigations

Investigation 1

Exponential Growth

In Investigation 1, students explore situations that involve repeated doubling, tripling, and quadrupling. Students are introduced to one of the essential features of many exponential patterns: rapid growth.

Students make and study tables and graphs for exponential situations, describe the patterns they see, and write equations for them, looking for a general form of an exponential equation. Students also compare and contrast linear and exponential patterns of growth.

Investigation 2

Examining Growth Patterns

Investigation 2 focuses on exponential relationships with y-intercepts greater than 1. The standard form of an exponential equation is $y = a(b)^x$. When $x = 0$, the equation becomes $y = a$ since $b^0 = 1$. Thus a, the coefficient of the exponential term, generally indicates the initial value of the exponentially growing quantity. This initial value is the y-value corresponding to $x = 0$, or the y-intercept. Each problem in the investigation presents information about an exponential pattern in a different form—in a verbal description, in an equation, and as a graph—helping students develop flexibility in moving among representations.

Investigation 3

Growth Factors and Growth Rates

In Investigation 3, students study non-whole-number growth factors other than 1 and relate these growth factors to *growth rates*. As an example, consider money invested at 6% annual interest. To find the amount of money for a given year, multiply the amount from the previous year by 1.06. The growth factor in this case is 1.06, while the growth rate is 6% (or 0.06). Students also explore how the growth rate and the initial value affect the growth pattern.

Investigation 4

Exponential Decay

Investigation 4 introduces students to *exponential decay*—patterns of change that exhibit successive, non-constant decreases rather than increases. These decreasing relationships are generated by repeated multiplication by factors between 0 and 1, called *decay factors*. Strategies for finding decay factors and initial population and for representing decay patterns are similar to those used for exponential growth patterns.

Investigation 5

Patterns With Exponents

Investigation 5 develops rules for operating with exponents. Students examine patterns among the ones digits of powers and use these patterns to predict ones digits for powers that would be tedious to find directly. Then, they look for relationships among numbers written in exponential form. This leads to the rules for operating on numerical expressions with exponents. Finally, students use graphing calculators to study the effects of the values of a and b on the graph of $y = a(b^x)$.

Mathematics Background

The basic goal in *Growing, Growing, Growing* is for students to learn to recognize situations, data patterns, and graphs that are modeled by exponential equations and to use tables, graphs, and equations to answer questions about exponential patterns. This unit is designed to introduce the topic and to give students a sound, intuitive foundation on which to build later.

Exponential Growth

An exponential pattern of change can often be recognized in a verbal description of a situation or in the pattern of change in a table of (x, y) values.

Suppose you offer one of your classes a reward for days on which everyone works diligently for the entire class period. At the start of the year, you put 1 cent in a party fund. You promise that on the first good-work day, you will contribute 2 cents; on the second good-work day, you will contribute 4 cents; and on each succeeding good-work day, you will double the reward of the previous good-work day.

Class Party Fund

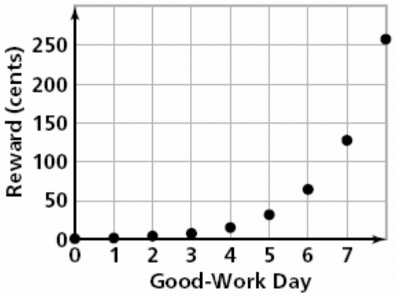
Good-Work Day	Reward (cents)
0 (start)	1
1	2
2	4
3	8
4	16
5	32
6	64
7	128
8	256

Growth Factor

For each good-work day, the monetary reward doubles. You multiply the previous reward by 2 to get the new reward. This constant factor can also be obtained by dividing each successive y-value by the previous y-value: $\frac{2}{1} = 2, \frac{4}{2} = 2$, and so on. This ratio is called the *growth factor* of the pattern.

The exponential growth in rewards for good-work days in the example can be represented in a graph. The increasing rate of growth is reflected in the upward curve of the plotted points.

Class Party Fund



Exponential Equations

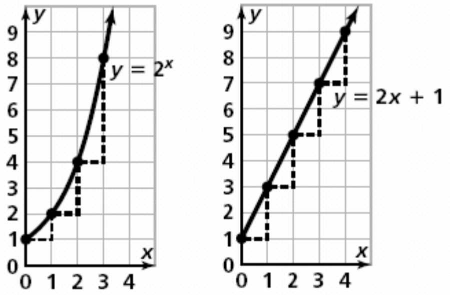
Examining the growth pattern in the class reward leads to an equation.

Day	Calculation	Reward (cents)
0	1	1
1	$1 \times 2 = 2^1$	2
2	$1 \times 2 \times 2 = 2^2$	4
3	$1 \times 2 \times 2 \times 2 = 2^3$	8
⋮	⋮	⋮
6	$1 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^6$	64
⋮	⋮	⋮
n	$1 \times 2 \times 2 \times \ldots \times 2 = 2^n$	2^n

This growth pattern can be summarized in symbolic form using exponents. For example, the reward on the tenth good-work day can be expressed as $1 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^{10}$.

On the nth good-work day, the reward r will be $r = 2^n$. Because the independent variable in this pattern appears as an exponent, the growth pattern is called *exponential*. The growth factor is the *base,* 2. The *exponent n* tells the number of times the 2 is a factor.

It is important to distinguish between a constant growth *factor* (multiplicative), as just illustrated in an exponential pattern, and the constant *additive* pattern in linear relationships. In the graphs of $y = 2^x$ and $y = 2x + 1$ below, the horizontal change is the same. On the graph of $y = 2x + 1$, the vertical change is a constant. On the graph of $y = 2^x$, the vertical change increases by a multiple of the growth factor as the graph rises.



In a linear situation with equation $y = mx + b$, as x increases by 1, the value m is added to get the new y-value. The difference between any two consecutive terms in a linear relationship is that constant additive change m. In contrast, an exponential growth pattern, $y = a(b)^x$, may increase slowly at first but grows at an increasing rate because its growth is multiplicative. The growth factor is b.

y-Intercept or Initial Value

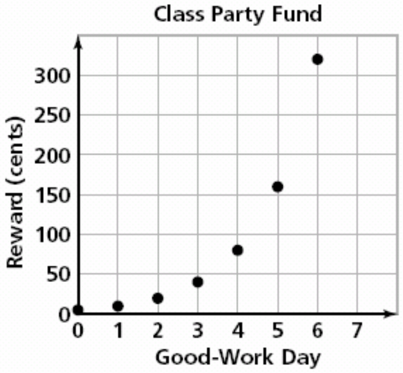
In the preceding example, the y-intercept was (0, 1). The following example illustrates a y-intercept that is not equal to 1.

The class party fund began with only 1 cent. That might strike students as a tiny seed for the fund, so suppose you made a more generous initial offer of 5 cents. The table and graph for this new reward scheme follow; an equation (with the usual variable names, x and y) to represent it would be $y = 5(2^x)$.

Note that the growth factor is still 2. The reward on any given good-work day is twice that of the previous day. The reward is five times the reward for the same day in the original scheme, and the new starting amount is reflected in the equation by multiplying the original reward by 5. The equation is $y = 5(2^x)$. In the standard form for exponential equations, $y = a(b^x)$, a is the y-intercept, and b is the growth factor.

Class Party Fund

Good-Work Day	Reward (cents)
0 (start)	5
1	10
2	20
3	40
4	80
5	160
6	320



Note that students sometimes refer to the y-intercept (or initial value) as the "starting point." For exponential situations, they use the starting point and the growth factor to generate a table by multiplying the previous term by a constant factor. For a linear situation, they use the starting point (y-intercept) and the constant rate of change to generate a table by adding a constant amount to the previous term.

Growth Rates

A growth rate is different from, but related to, a growth factor. The following example will illustrate the connection between the two concepts.

Suppose you put 5 cents in the Class Party Fund and then increase the reward by 8% for each succeeding good-work day. Figure 1 shows the calculations required to find the reward for each day.

The 8% increase is the growth rate. By examining the pattern in the reward column, you can see that the growth factor is 1.08. (Divide each reward value by the previous reward value.) The equation for the relationship between the work day n and reward r is: $r = 5(1.08)^n$

Another way to find the growth factor is to apply the distributive property at each stage of the calculations:

Day 1: $5 + 0.08 \times 5 = 5(1 + 0.08) = 5 \times 1.08$

Day 2: $(5 \times 1.08) + 0.08 \times (5 \times 1.08)$
$= (5 \times 1.08)(1 + 0.08)$
$= (5 \times 1.08)(1.08)$
$= 5 \times (1.08)^2$

Continuing this process gives

Day n: $5 \times (1.08)^n$

In general, a growth rate of r is associated with a growth factor of $(1 + r)$. Similarly, if the growth factor is f, then the growth rate is $(f - 1)$. Growth rates are often expressed as percents.

Figure 1

Good-Work Day	Calculation	Reward (cents)
0 (start)	5	5
1	5 + 0.08 × 5	5.4
2	5.4 + 0.08 × 5.4	5.832
3	5.832 + 0.08 × 5.832	6.29856
4	6.29856 + 0.08 × 6.29856	6.8024448
5	6.8024448 + 0.08 × 6.8024448	7.346640384
6	7.346640384 + 0.08 × 7.346640384	7.934371615
⋮	⋮	⋮
n		$5(1.08)^n$

Exponential Decay

Exponential models also describe patterns in which the value of a dependent variable decreases as time passes. In this case, the constant multiplicative factor is referred to as the decay factor. Decay factors work just like growth factors, only they result in decreasing relationships because they are between 0 and 1.

Suppose another teacher offers a different incentive for good-work days. At the start of the school year, the teacher puts $50 in a class party fund. For each day the class does not work diligently, she cuts the party fund in half.

As the days pass, the class party fund will decrease in the pattern shown in the following table. The exponential decay pattern is also represented in the graph. The plotted points begin at (0, 50) and drop from left to right. Notice that, although half the amount is removed at each stage, the amount removed each time decreases.

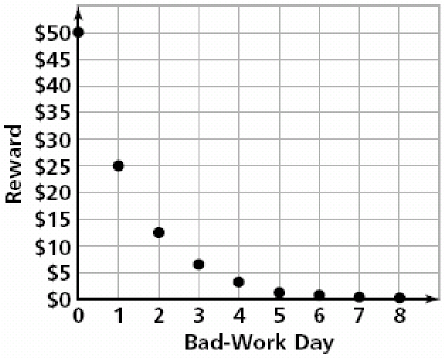
Class Party Fund

Bad-Work Day	Reward
0 (start)	$50.00
1	$25.00
2	$12.50
3	$6.25
4	$3.13
5	$1.56
6	$0.78
7	$0.39
8	$0.20

The decay factor for this exponential decay pattern is $\frac{1}{2}$. The amount in the party fund f after n bad-work days is given by the equation $f = 50(\frac{1}{2})^n$.

This exponential model is similar to that for exponential growth except that the repeating factor, the base, is a positive number less than 1.

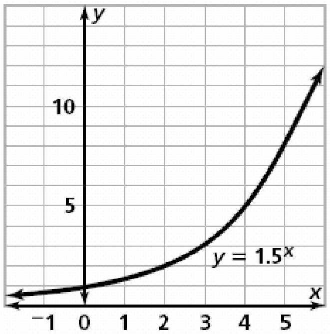
Class Party Fund



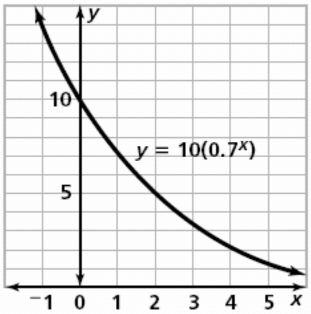
Graphs of Exponential Relationships

The basic patterns of exponential growth and exponential decay involve change from one point in time to the next by some constant factor. For growth, the change factor is a number greater than 1 and the graph curves upward from left to right. For decay, the change factor is between 0 and 1 and the graph curves downward from left to right, approaching the x-axis but never reaching it.

Exponential relationships can also be defined for negative and non-integer values of the exponent. The related graphs are *continuous curves* (rather than graphs of plotted points) with shapes similar to those shown.

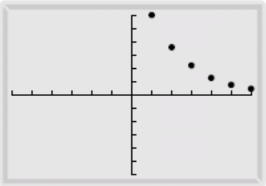


Students will recall from the unit *Variables and Patterns* the difference between graphs where the dots are connected and those where the dots are not connected. In this unit, that distinction is not an important one. In fact, it is often useful to connect the dots to highlight a pattern. In such a case, though, it is important to remember that the points corresponding to non-integer values of x may not arise from the data of the problem at hand.

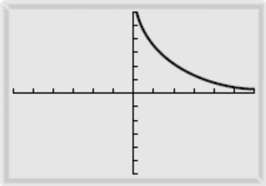


The focus of this unit is primarily on positive integer exponents, so the graphs will generally occur in the first quadrant. Depending on the situation, the graph will show discrete points (with or without a curve through the points) or a continuous curve.

It is illuminating to occasionally ask students what information the continuous graphs produced by graphing calculators communicate at non-integer points. For example, in many contexts where time is the independent variable, the corresponding y-values have quite natural interpretations.



Graph of plotted points



Graph of related equation

The formal definition of exponential expressions for non-integer exponents is delayed until a later course.

Tables of Exponential Relationships: Recursive or Iterative Processes

Students usually generate each value in their tables based on the previous value. Either they add a constant to the previous value (in the case of linear relationships) or they multiply the previous value by a constant (in the case of exponential relationships). This process of generating a value from a previous value is called *recursion* or *iteration*.

Equivalence of Two Forms of Exponential Equations

In the first investigation of this unit, students may end up writing two different, but equivalent, exponential equations for this situation:

A king places 1 ruba on the first square of a chessboard, 2 rubas on the second square, 4 on the third square, 8 on the fourth square, and so on, until he has covered all 64 squares. Each square has twice as many rubas as the previous square.

By examining the patterns in a table, students write an equation for the number of rubas r on square n.

Square Number	Number of Rubas
1	1
2	2
3	4
4	8
5	16

Some students will note that the number of rubas on a given square n is a product of $n - 1$ twos and write $r = (2^{n-1})$.

Other students will reverse the pattern and find the number of rubas on "square 0" by dividing by the number of rubas on square 1 by 2. This gives them the y-intercept, $\frac{1}{2}$. (Square 0 has no meaning in this context, but many students find it useful to use the y-intercept as a starting point when they write an equation.) They then note that the number of rubas on square n is half the product of n twos, and write $r = \frac{1}{2}(2^n)$.

It is important for students to recognize that the two forms, $r = (2^{n-1})$ and $r = \frac{1}{2}(2^n)$, are equivalent. They can verify the equivalence by generating tables or graphs.

The following is a general argument for why b^{x-1} is equivalent to $\frac{1}{b}(b^x)$ for any value of b. (In the example above, $b = 2$.)

The equation $y = (b^{x-1})$ is equivalent to $y = (b^x) \times b^{-1}$. Because $b^{-1} = \frac{1}{b}$, this is equivalent to $y = \frac{1}{b}(b^x)$.

This argument is provided for your information. Students do not need to understand it at this point in their development.

Logarithms

Understanding logarithms is *not* a goal of this unit. Logarithms are not mentioned to students. Nonetheless, there are several questions in the unit that push students to think about the ideas behind logarithms. For example, in Problem 1.1, students examine a situation in which a sheet of paper is cut in half, the resulting two pieces are stacked and the stack is cut in half, the resulting four pieces are stacked and the stack is cut in half, and so on. Students write the equation $y = 2^x$ to describe the relationship between the number of cuts x and the number of pieces of paper y. In one question, they are asked how many cuts it would take to create at least 500 pieces of paper. The answer is the solution to $500 = 2^x$.

Students will and should estimate the solution by using a guess-and-check method or by generating a calculator table or graph. In high school, they will learn to use logarithms to solve such an equation exactly.

Logarithmic functions are the inverses of exponential functions, just as division is the inverse of multiplication. We can rewrite the preceding equation as:

$x = \log_2 500$

Some calculators can compute logarithms using any base (here the base is 2). Most scientific calculators are limited to the bases 10 and e. In this case, the solution is more complicated:

$500 = 2^x$

$\log_{10}(500) = \log_{10}(2^x)$

$\log_{10}(500) = x\log_{10}(2)$ (by laws of exponents)

So, $x = \frac{\log_{10} 500}{\log_{10} 2}$.

Remember that all of this is far beyond what we ask students to do in this unit.

Not all problems can be solved by applying standard algorithms. For example, there is no algebraic technique for finding the point of intersection of an exponential relationship and a linear relationship, such as $y = (2^x)$ and $y = 5x + 15$. The best we can do is estimate the intersection point of the graphs of the equations. Sophisticated estimation techniques exist, but it is impossible to solve such a problem directly.

Rules of Exponents

Students begin to develop understanding of the rules of exponents by examining patterns in the powers charts for the first 10 whole numbers. (See Figure 2.)

Students observe that several numbers occur more than once in the table. For example, 64 occurs as 2^6, 8^2, and 4^3. By examining the multiplicative structure of the bases, they find that $8^2 = (2 \times 2 \times 2)^2 = (2^3)^2 = 2^6$. After several examples, students conjecture that $(b^m)^n = b^{mn}$.

Students multiply pairs of numbers in the same column, say 9 and 27 in column 3: $9 \times 27 = 243$ or $3^2 \times 3^3 = 3^5$. After looking at several examples, students conjecture that $(b^m)(b^n) = b^{m+n}$.

Students also multiply pairs of numbers in the same row, say 4 and 25 in row 2:
$4 \times 25 = 2^2 \times 5^2 = (2 \times 5)^2 = 10^2 = 100$.
The general pattern is $(a^m b^m) = (ab)^m$. Similar explorations lead to the rule $\frac{a^m}{a^n} = a^{m-n}$.

Students also note that the ones digits for the powers repeat in cycles of 1, 2, or 4 and apply this observation to predict ones digits of powers and to estimate the value of exponential expressions.

Figure 2

Powers Table

x	1^x	2^x	3^x	4^x	5^x	6^x	7^x	8^x	9^x	10^x
1	1	2	3	4	5	6	7	8	9	10
2	1	4	9	16	25	36	49	64	81	100
3	1	8	27	64	125	216	343	512	729	1,000
4	1	16	81	256	625	1,296	2,401	4,096	6,561	10,000
5	1	32	243	1,024	3,125	7,776	16,807	32,768	59,049	100,000
6	1	64	729	4,096	15,625	46,656	117,649	262,144	531,441	1,000,000
7	1	128	2,187	16,384	78,125	279,936	823,543	2,097,152	4,782,969	10,000,000
8	1	256	6,561	65,536	390,625	1,679,616	5,764,801	16,777,216	43,046,721	100,000,000
Ones Digits of Powers	1	2, 4, 8, 6	3, 9, 7, 1	4, 6	5	6	7, 9, 3, 1	8, 4, 2, 6	9, 1	0