# Unit Introduction

# Frogs, Fleas, and Painted Cubes

## Quadratic Relationships

### Goals of the Unit

- Recognize the patterns of change for quadratic relationships
- Write equations for quadratic relationships represented in tables, graphs and problem situations
- Connect quadratic equations to the patterns in tables and graphs of quadratic relationships
- Use a quadratic equation to identify the maximum or minimum value, the $x$- and $y$-intercepts, and other important features of the graph of the equation
- Recognize equivalent quadratic expressions
- Use the Distributive Property to write equivalent quadratic expressions in factored and expanded form
- Use tables, graphs, and equations of quadratic relationships to solve problems in a variety of situations from geometry, science, and business
- Compare properties of quadratic, linear, and exponential relationships

### Developing Students' Mathematical Habits

As students work on the problems in this unit, they learn to ask themselves questions about problem situations that involve non-linear relationships:

- *What are the variables?*
- *How can I recognize whether the relationship between the variables is quadratic?*
- *What equation models a quadratic relationship in the table, graph, or problem context?*
- *How can I answer questions about the situation by studying a table, graph, or equation of the quadratic relationship?*

# Mathematics of the Unit

Pearson Prentice Hall Professional Development

## Overview

The fundamental goal of the *Connected Mathematics* algebra units is to develop students' abilities to describe and analyze relationships between variables. This goal encompasses gaining an understanding of and skill in using tabular (tables), graphic (graphs), and symbolic (equations) modes of representing and reasoning about such relationships. It also includes developing familiarity with several of the most common patterns of variation.

In the grade 7 units *Variables and Patterns* and *Moving Straight Ahead,* students were introduced to basic concepts and representations of algebra, focusing on linear functions. In the grade 8 unit *Thinking with Mathematical Models*, they revisited linear models (as in the bridge-thickness experiment) and investigated several examples of inverse variation (as in the bridge-length experiment and distance-rate-time problems). In *Growing, Growing, Growing*, students explored exponential models. In *Frogs, Fleas, and Painted Cubes*, the focus switches to a nonlinear polynomial relationship—the second-degree polynomial, or the quadratic function. By investigating a variety of problem situations, students learn about the characteristics of quadratic relationships, as represented in tables, graphs, and equations.

The relationships in this unit are all functions. Through their work with these relationships, students continue to develop an intuitive understanding of the term *function*. For example, the height of a jumping frog or flea is a function of, or depends on, the amount of time the jumper has been in the air. The area of a rectangle with a fixed perimeter is a function of, or depends on, its side length. The number of painted faces on the small cubes that comprise a larger cube is a function of, or depends on, the edge length of the large cube. Some examples of cubic functions, or third-degree polynomials, are included for comparison and contrast.

Each of the three types of representations of quadratic functions—graphs, tables, and equations—gives important information about the situation being modeled. Students look for commonalities and differences among the three representations, and use them to answer questions.

Questions related to the graphs of quadratic relationships emphasize their shape (parabolas), the location and interpretation of intercepts and lines of symmetry, and the presence and location of maximum or minimum points.

Questions related to tables of quadratic relationships focus on patterns in the rate of change in $y$-value as the $x$-value increases (or decreases), and contrast those patterns with the constant rate of change that characterizes linear functions. Patterns in tables can also reveal to students the locations of intercepts and maximum or minimum points.

Questions about equations of quadratic relationships focus on connecting the symbolic form to tabular and graphic forms, and determining significant intercepts. The problems in the unit lead to both the factored and expanded forms of quadratic equations. Through their work, students will discover the equivalence of the two forms and learn that each is convenient for gathering different types of information. While there is some simple symbol manipulation of quadratic expressions, students are not expected to master the traditional symbol manipulation procedures associated with a traditional polynomial algebra course. In the next two algebra units, students will have additional opportunities to determine and use factored and expanded forms of quadratic equations and to solve quadratic equations by factoring the symbolic form of the equation. In this unit, they use graphs and tables to solve quadratic equations by finding the $x$-intercepts.

## Summary of Investigations

### Investigation 1

### Introduction to Quadratic Relationships

Students look at the area of rectangles with a fixed perimeter to discover that within the family of rectangles with a fixed perimeter, a square has the greatest (maximum) area. Using tables and graphs to represent the data, they learn to recognize the shape of a quadratic function (a parabola), connect the shape to patterns in the table, and describe special features of the quadratic relationship, such as intercepts and maximum points. Students write an equation for the relationship between the area and length of rectangles with a fixed perimeter, $P$. The relationship is a quadratic function of the length, $\ell$, or width, $w$.

$A = \ell(\frac{P}{2} - \ell)$ or $A = w(\frac{P}{2} - w)$

## Investigation 2

### Quadratic Expressions

Students investigate how increasing one dimension of a square (with sides of length $n$) by 2 and decreasing the other dimension by 2 affects the area. The area of the rectangle is $(n + 2)(n - 2)$. Students discover that the area of the square is always 4 units greater than the area of the rectangle, so the area of the rectangle is $n^2 - 4$. Students explore a visual representation of the Distributive Property. They discover that they can represent quadratic relationships as the product of two linear expressions, called *factored form,* or as the sum of one or more terms, called *expanded form*. For example, if one dimension of a square of length $n$ is increased by 2 and the other dimension is increased by 3, then the area of the rectangle can be written as $(n + 2)(n + 3)$ or as $n^2 + 5n + 6$. Students use the Distributive Property to write quadratic expressions in equivalent forms from expanded form to factored form and from factored form to expanded form. Students make connections among tables, graphs, and equations to learn which form of a quadratic equation provides specific information about the relationship.

## Investigation 3

### Quadratic Patterns of Change

Students investigate quadratic relationships in sequences of triangular, square, and rectangular numbers. They discover that the model for the number of handshakes is the same as for other problems, including triangular numbers. Students consider variations on the handshake problem, exploring questions such as: when two athletic teams each with $n$ members line up to shake hands after a game, how many handshakes take place? How many handshakes take place among teams with different numbers of players? What if members of a single $n$-member team exchange "high fives" after a win? By comparing and analyzing patterns in tables, graphs, and equations, they express these quadratic relationships with equivalent expressions and predict whether variations in the handshake problem are quadratic.

## Investigation 4

### What Is a Quadratic Function?

This investigation uses the classic projectile motion problems to extend students' understanding of quadratic polynomials and their graphs. Students find significant intercepts, and maximum or minimum points, from quadratic relationships given in standard form. Investigating patterns of change in quadratic equations more closely, students discover that differences between consecutive $y$ values, called *first differences,* are constant for linear relationships, and that *second differences* are constant for quadratic relationships. They also discover that in cubic relationships, *third differences* are constant.

## Mathematics Background

The word "quadratic" can be misleading, because it seems to imply a connection to the number four. The prefix *quad* relates to the classic problem of trying to find a square with the same area as a given circle. This is known as finding the *quadrature* of the circle. So the name refers to finding the area, $x^2$, of a square with a side length of $x$.

#### Representing Quadratic Functions with Equations

Quadratic relationships are defined as relationships of the form $y = ax^2 + bx + c$, in which $a$, $b$, and $c$ are constants and $a \neq 0$. This form of the equation is called *expanded form*. This definition emphasizes that the independent variable is raised to the second power. While this is a useful definition, it is also important to understand the *factored form* of quadratic equations.

Quadratic functions arise from situations with an underlying multiplicative relationship, such as the area of rectangles. So quadratic relationships can also be defined as functions whose $y$-value is the product of two linear factors—the form

$y = (ax + c)(bx + d)$, where $a \neq 0$ and $b \neq 0$. The power of this form is that it connects polynomials to products of linear factors. For example, you can also represent $y = 2x^2 + 3x - 2$ as $y = (2x - 1)(x + 2)$. The factored form can help you determine $x$-intercepts and the location of the maximum and minimum points.

The first context students explore is finding the maximum area for a rectangle with a fixed perimeter. This situation allows students to look at characteristic patterns of change for quadratic functions in a table, graph, and equation.

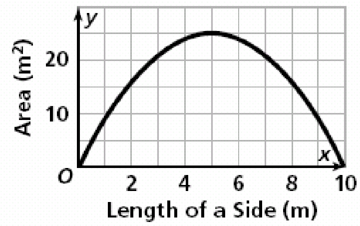
If the perimeter of a rectangle is 20 meters, then the area $A$ of the rectangle can be represented as $A = \ell w$, where $\ell$ is the length and $w$ is the width. Since $2(\ell + w) = 20$ or $\ell + w = 10$, $w$ can be written as $(10 - \ell)$ and the area can be written in terms of one of its dimensions, $A = \ell(10 - \ell)$.

Examining the table, graph, and equation for this relationship provides an introduction to quadratic functions.

**Rectangles With Perimeters of 20 m**

| Length of Base | Width | Area |
|---|---|---|
| 0 | 10 | 0 |
| 1 | 9 | 9 |
| 2 | 8 | 16 |
| 3 | 7 | 21 |
| 4 | 6 | 24 |
| 5 | 5 | 25 |
| 6 | 4 | 24 |
| 7 | 3 | 21 |
| 8 | 2 | 16 |
| 9 | 1 | 9 |
| 10 | 0 | 0 |

**Areas of Rectangles With Perimeter 20 Meters**




This situation provides an opportunity to explore the pattern of change in the table and how it relates to the graph and equation. As length increases by 1, area increases to a certain point, and then at a fixed point (maximum), area starts to decrease. The maximum area occurs halfway between the $x$-intercepts [(0, 0) and (10, 0)], at the point (5, 25), on the line of symmetry. The line of symmetry is a vertical line through the maximum point; it divides the graph into two congruent parts. So the shape of the rectangle with a maximum area of 25 square units is a square with side lengths of 5 meters.

### Representing Quadratic Patterns of Change with Tables

In this unit, students often make tables to represent quadratic equations. Patterns of change in quadratic relationships are most readily observed in tables.

In linear relationships, as the $x$-values increase by one in the "*first" differences*—the difference between consecutive $y$-values—are constant, indicating a constant rate of change. In quadratic relationships, "second" differences—the differences between successive first differences—are constant. (Figure 1)

First differences represent the rate at which $y$ is changing with respect to $x$. That is, the first difference gives the change in $y$-values between $x$ and $x + 1$. Second difference indicates the rate at which the first difference is changing. If all of

**Figure 1**

$y = 6(x - 2)^2$

| $x$ | $y$ |
|---|---|
| 0 | 24 |
| 1 | 6 |
| 2 | 0 |
| 3 | 6 |
| 4 | 24 |
| 5 | 54 |

First Differences

- $6 - 24 = -18$
- $0 - 6 = -6$
- $6 - 0 = 6$
- $24 - 6 = 18$
- $54 - 24 = 30$

Second Differences

- $-6 - (-18) = 12$
- $6 - (-6) = 12$
- $18 - 6 = 12$
- $30 - 18 = 12$

the second differences are the same, then the relationship is quadratic.

### Connecting Patterns of Change to Calculus

Finding successive differences of polynomials relates to derivatives in calculus. The first derivative, $y'$, of $y = ax^2 + bx + c$ is $y' = 2ax + b$, which means that this rate is still dependent on $x$ and changes with $x$. The second derivative is $2a$, which is no longer dependent on $x$ but is constant. For linear functions, the first derivative is a constant.

The following argument shows why the second difference for quadratic relationships is $2a$.

For quadratic equations of the form $y = ax^2 + bx + c$, the second difference between two successive values of $y$ is constant and it is equal to $2a$. This can be seen by picking three successive values for $x$.

| $x$ | $ax^2 + bx + c$ | first difference | second difference |
|---|---|---|---|
| 0 | $c$ | | |
| | | $a + b$ | |
| 1 | $a + b + c$ | | $2a$ |
| | | $3a + b$ | |
| 2 | $4a + 2b + c$ | | |

Note this argument will work for any three consecutive values of $x$.

One way students can detect whether a pattern is quadratic is by calculating second differences.

### Extending Patterns of Change to Cubic Functions or Polynomial Functions

In cubic functions, such as $y = (x - 2)^3$, students make another interesting discovery—third differences are constant. This is a characteristic of cubic relationships, also called third-degree polynomials.

| $x$ | $y$ | First Differences | Second Differences | Third Differences |
|---|---|---|---|---|
| 0 | −8 | | | |
| | | 7 | | |
| 1 | −1 | | −6 | |
| | | 1 | | 6 |
| 2 | 0 | | 0 | |
| | | 1 | | 6 |
| 3 | 1 | | 6 | |
| | | 7 | | 6 |
| 4 | 8 | | 12 | |
| | | 19 | | 6 |
| 5 | 27 | | 18 | |
| | | 37 | | |
| 6 | 64 | | | |

Similar relationships hold for other polynomials. For example, fourth-degree polynomials have fourth-level differences constant, and so on. Polynomial relationships with different degrees all have characteristic graphs and tabular patterns.

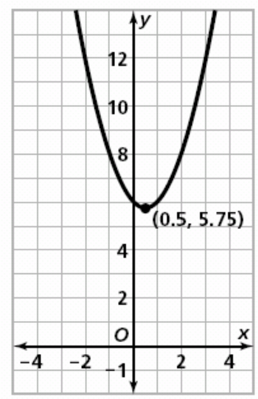
### Representing Quadratic Patterns of Change with Graphs

The values of $a$, $b$, and $c$ in the general equation $y = ax^2 + bx + c$ affect the shape, orientation, and location of the graph of a quadratic function, a parabola.

#### Maximum/Minimum Points

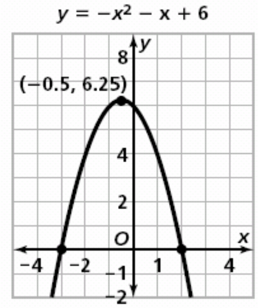
If the parameter $a$ (the coefficient of the $x^2$ term) in a quadratic is positive, the curve opens upward and has a minimum point, as shown below.

$y = x^2 - x + 6$



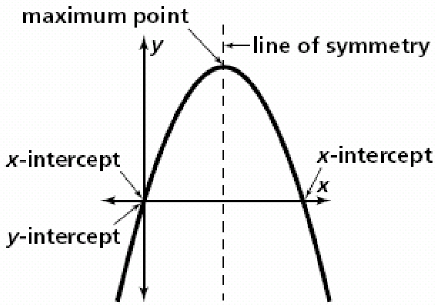

If $a$ is negative, the curve opens downward and has a maximum point, as shown below.

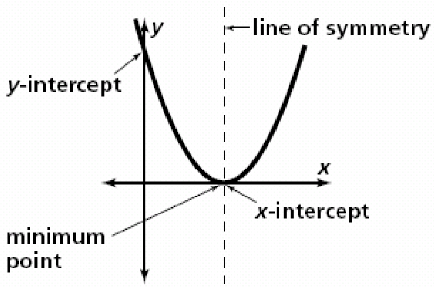
$y = -x^2 - x + 6$




#### The Line of Symmetry

The maximum or minimum point of the graph of a quadratic function (parabola) is called the *vertex*. The vertex lies on the vertical *line of symmetry* that separates the parabola into halves that are mirror images. The vertex is located halfway between the *x-intercepts*, if the $x$-intercepts exist. The *y-intercept* is the point where the parabola crosses the $y$-axis.




To find the maximum or minimum points and the intercepts of a parabola, students can make a table of values or trace the parabola on their calculators. More sophisticated methods for locating these features are outlined later. It is important for students to understand that a quadratic relationship is the result of multiplying two linear factors—factors in which the variable is raised to the first power.

**Note:** In this unit, when writing a quadratic expression in factored form is called for, the quadratic expression is factorable over rational numbers. However, not all quadratic expressions are factorable over rational numbers. For example, $-x^2 - x + 6$ *can* be written as a product of linear factors with rational coefficients $(-x + 2)(x + 3)$ while $x^2 - x + 6$ can *not*. In future algebra classes students learn other strategies for working with quadratic expressions such as these which are factorable over real numbers. The real numbers are the union of rational and imaginary numbers.

**$x$-Intercepts**

One way to find the $x$-intercepts is to set $y$ equal to 0 and solve for $x$. For example, to find the $x$-intercepts of the equation $y = -x^2 - x + 6$, set $y$ equal to 0 and solve for $x$.

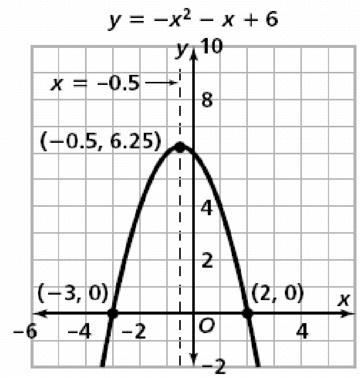
$$0 = -x^2 - x + 6$$
$$0 = x^2 + x - 6$$
$$0 = (x + 3)(x - 2)$$

Note that the expression is now in factored form. $x = -3$ or 2 are both solutions of this equation. So the $x$-intercepts are the points $(-3, 0)$ and $(2, 0)$. The vertex, which is a maximum point in this case, is be located halfway between the $x$-values at $-0.5$. To find the $y$-value of the vertex, substitute $-0.5$ for $x$ in the equation and solve for $y$.

$$y = -(-0.5)^2 - (-0.5) + 6 = 6.25$$

The maximum point is $(-0.5, 6.25)$. The equation of the line of symmetry is $x = -0.5$.

Here is a graph of the equation:




Most of the time it is not possible to factor a quadratic equation. You can also solve quadratic equations by applying the *quadratic formula*.

The quadratic formula is:

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$

For example, applying the quadratic formula to the equation $y = -2x^2 + 3x + 4$ gives the following:

$$0 = -2x^2 + 3x + 4$$
$$x = \frac{-3 \pm \sqrt{(9 - 4(-2)(4))}}{2(-2)}$$
$$x = -0.85 \text{ or } 2.35.$$

In general, the $x$-intercepts are located at

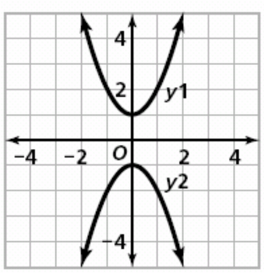
$$\left(\frac{-b + \sqrt{(b^2 - 4ac)}}{2a}, 0\right) \text{ and}$$

$$\left(\frac{-b - \sqrt{(b^2 - 4ac)}}{2a}, 0\right).$$ The $x$ value of

the vertex lies halfway between these $x$ values at $\frac{-b}{2a}$. This can be substituted into the original equation to determine the corresponding $y$ value.

In this unit, students do not need to know how to use the quadratic formula. They can use a table or trace a graph on the calculator to read the maximum, and the intercepts.

The expression under the radical sign, $b^2 - 4ac$, is called the **discriminant**. Many times the discriminant is an irrational number such as $\sqrt{2}$. In this case, the quadratic expression is not factorable. If the discriminant is negative, then the situation yields the square root of a negative number, which does not exist in the set of real numbers. The square roots of negative numbers are called *imaginary numbers*. The set of *complex numbers* consist of all sums of real numbers and imaginary numbers. If the $x$-intercepts are imaginary, then the graphs do not intersect the $x$-axis. Two examples and their graphs are $y_1 = x^2 + 1$ and $y_2 = -x^2 - 1$.



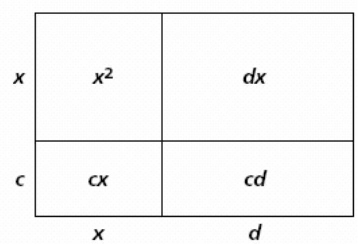

This unit includes only quadratic expressions that can be easily factored, so that students can develop an understanding of equivalent forms for a quadratic expression and the information that each expression offers.

### The Distributive Property and Equivalent Quadratic Expressions

Students explore quadratic equations in both factored form and expanded form. They learn that quadratic expressions are either the product of two linear factors, such as $y = (x + 1)(x - 1)$, or the sum or difference of terms that include an $x^2$ term and no higher exponent of $x$, such as $y = 2x^2 + 6x - 10$.

If the expression is in factored form, then it must contain exactly two factors, each with the variable $x$ raised to the first power. For example, $2x(x + 3)$ is a quadratic expression, but $2(x + 3)$ is not a quadratic expression. The factor 2 is linear, but it does not contain a variable raised to the first power. The two factors of a quadratic expression may also be called *binomial expressions* or binomials. A binomial is an expression with two terms. This unit focuses on linear factors that contain the variable raised to the first power, in order to emphasize the connection between linear and quadratic functions.

While students do a bit of factoring and multiplying two binomials, this unit emphasizes finding equivalent expressions. For example, you can think of the area of the rectangle below as the product of two linear expressions, the result of multiplying the width by the length. This produces the *factored form* of a quadratic expression. You can also think of the area as the sum of the areas of the subparts of the rectangle. This generates the *expanded form* of a quadratic expression.




$A = (x + c)(x + d)$ factored form
$A = x^2 + cx + dx + cd$ expanded form

In *Say It With Symbols*, students address multiplying binomials and factoring quadratic expressions in greater detail.

### A Note on Terminology

To introduce quadratic expressions, this unit uses the term *expanded form* instead of *standard form* to help students remember that expanded form represents the sum of the areas of the smaller rectangles that compose the large rectangle. Also, the word *standard* implies a preferred form. Students should be able to build on prior learning by identifying factored and expanded forms as examples of the distributive property, as discussed in earlier units: *Accentuate the Negative, Moving Straight Ahead,* and *Thinking with Mathematical Models.*

This unit also briefly exposes students to cubic equations, which are either the product of three linear factors, such as $y = (x - 2)(x - 2)(x - 2)$, or a sum or difference of terms that include an $x^3$ term and no higher exponent of $x$, such as $y = x^3 - 6x^2 + 12x - 8$.

### Other Contexts for Quadratic Functions

#### Counting Handshakes

The classic handshake problem and variations on it offer an interesting context for students to explore quadratic functions.

> *When a team of* n *members exchanges high fives at the end of a game, how many high fives take place?*

Students can draw diagrams, look for patterns in tables, or use reasoning, as follows:

- Each person high fives with $(n - 1)$ people, so there are $n(n - 1)$ high fives. But this counts each high five twice, so we must divide by two. The number of high fives, $h$, is $h = \frac{n(n-1)}{2}$.
- The first person high fives $n - 1$ people; the second person high fives $n - 2$ people, the third person high fives $n - 3$ people, and so on. The total number of high fives, $h$, is $h = 1 + 2 + 3 + \ldots + n - 1$.

#### Sum of the First *n* Counting Numbers

If both examples of reasoning above are valid, then we can claim that $1 + 2 + 3 + \ldots + n - 1 = \frac{n(n-1)}{2}$. This gives a formula for finding the sum of the first $n - 1$ counting numbers. If we add $n$ to both sides, we have a formula for finding the sum of the first $n$ counting numbers:

$$1 + 2 + 3 + \ldots + n - 1 + n = \frac{n(n-1)}{2} + n = \frac{n(n+1)}{2}.$$

In the ACE for Investigation 3, there is a Did You Know? feature about Carl Gauss, a famous mathematician who discovered this formula when he was a young student. Gauss' method works for any arithmetic sequence. An *arithmetic sequence* is a sequence of numbers in which the difference between any two consecutive terms is constant. Below is a generalization of his method, which works for any arithmetic sequence.

#### Generalization of Gauss's Method

The sequence of counting numbers,
1, 2, 3, 4, 5, 6, . . .
is an arithmetic sequence (the difference is 1); so is the sequence 60, 65, 70, 75, 80, 85, . . . (the difference is 5). Notice that the sequence does not have to start with 1.

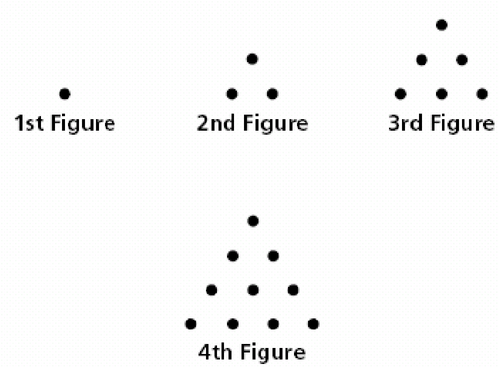
For example, here's how to find the sum of the odd whole numbers from 5 to 13:

$$5 + 7 + 9 + 11 + 13 = \frac{5}{2} \times (5 + 13) = 45$$

Another way to express the sum is $\frac{n}{2}$(first + last), where $n$ is the number of terms being added, "first" is the first number in the sequence, and "last" is the last number in the sequence.

#### Triangular Numbers

Triangular numbers are numbers that can be represented by a triangular array of dots:




You can think of each figure as the sum of the first $n$ whole numbers. For example, in the

3rd Figure, the number of dots is $1 + 2 + 3$ or 6. In the 4th Figure the number of dots is $1 + 2 + 3 + 4$ or 10. The sequence 1, 3, 6, 10, 15, 21, … represents triangular numbers. The equation for the *n*th triangular number, is $T_n = \frac{n(n + 1)}{2}$. It is similar to the equation for the number of high fives for a team of *n* members. The number of high fives for *n* people is actually the $(n - 1\text{th})$ triangular number.

It is interesting for students to observe that the equation $y = \frac{n(n - 1)}{2}$ could represent the number of high fives, *y*, that are exchanged among *n* people; or the $(n - 1\text{th})$ triangular number; or the sum of the first $(n - 1)$ counting numbers.

The first few contexts in Investigations 1–3 can be represented by equations of the form $y = x(x + a)$ or $y = x(a - x)$. In Investigation 4, students explore classic projectile problems that are represented by $y = ax^2 + bx + c$.

**Equations Modeling Projectile Motion**

Equations such as $h = -16t^2 + 64t + 6$ model one-dimensional projectile motion. They are based on the principles of mechanics. These equations assume that motion occurs in a vacuum, but you can make reasonably accurate predictions about the motion for modest speeds in normal air. The general form is $h = -16t^2 + v_0t + h_0$, in which the coefficient $-16$ is determined by the Earth's gravitational pull on the flying body; $v_0$ is the initial velocity of the projectile, and $h_0$ is the initial height of the projectile.

In the problems involving basketball players, fleas, and frogs jumping, students may bring up the topic of the speed at which the jumpers jump. Speed is associated with change in distance—in this case, a change in height. The jumper's speed decreases en route to the maximum height and increases on the return trip to the ground. The change in speed is reflected in the tables by the increasingly smaller change in *y*-value as the *x*-value increases, until the maximum *y*-value is reached. At the top of the jump, the speed is 0 for an instant. After the maximum height is attained, *y*-values decrease by increasing increments, reflecting the change in speed as the jumper returns to the ground.

Patterns of change is a unifying theme for looking at linear, exponential, and quadratic functions in *Connected Mathematics*. As with linear and exponential relationships, students explore the patterns of change that characterize quadratic relationships and learn to recognize these patterns of change in tabular, graphical, or symbolic representations.