Unit Introduction

Comparing and Scaling

Ratio, Proportion, and Percent

Goals of the Unit

- Analyze comparison statements made about quantitative data
- Use ratios, fractions, differences, and percents to form comparison statements in a given situation, such as

  "What is the ratio of boys to girls in our class?"

  "What fraction of the class is going to the spring picnic?"

  "What percent of the girls play basketball?"

  "Which model of car has the best fuel economy?"
- Judge whether comparison statements make sense and are useful
- See how forms of comparison statements are related (for example, a percent and a fraction comparison)
- Make judgments about which statements are most informative or best reflect a particular point of view
- Decide when the most informative comparison is the difference between two quantities and when it is a ratio between pairs of quantities
- Scale a ratio, rate, or fraction to make a larger or smaller object or population with the same relative characteristics as the original
- Represent related data in tables
- Look for patterns in tables that will allow predictions to be made beyond the tables
- Write an equation to represent the pattern in a table of related variables
- Apply proportional reasoning to solve for the unknown part when one part of two equal ratios is unknown
- Set up and solve proportions that arise in applications
- Recognize that constant growth in a table is related to proportional situations
- Connect a unit rate to the equation describing a situation

Developing Students' Mathematical Habits

The overall goal of the *Connected Mathematics* curriculum is to help students develop sound mathematical habits. Through their work in this and other number units, students learn important questions to ask themselves about situations that can be modeled mathematically.

- *When quantities have different measurements, how can they be compared?*
- *When can a comparison be made by subtraction?*
- *When can division be used?*
- *Why is a ratio a good means of comparison?*
- *How can a ratio be scaled up or down?*
- *How does rounding affect the numbers used in a ratio?*
- *What is the relationship between ratios and similar figures?*
- *How can ratios be used in daily life to find unknown quantities or inaccessible measurements?*
- *How can we use proportions to solve problems?*

Mathematics of the Unit

Pearson Prentice Hall Professional Development

Overview

Although quantitative problems can be solved simply by counting members of a set or by measuring, it is often necessary to make decisions that involve comparisons of counts or measurements. The basic step in this kind of thinking is developed in elementary grades when such comparisons are decided by finding which number is greater. However, more useful reasoning often requires more careful comparison—explaining how much greater one number is than another, not in an absolute sense, but in a relative sense. There are many standard ways to make such comparisons (for example, fractions, ratios, rates, differences, and percents). One of the fundamental goals of school mathematics, especially middle-grades mathematics, is to help students develop flexible understanding, skill, and disposition in using strategies for comparing quantities. This goal runs throughout the Problems, ACE, and Reflections of this unit. The unit confronts students with a series of mathematical tasks that encourage them to make decisions about the quantities relevant to each task, how those quantities can be compared most usefully, and what information is provided by various quantitative comparisons.

The second major theme of this unit, as the title suggests, is *scaling*. In its most familiar sense, scaling suggests making something bigger or smaller, but similar in key respects to an original. Ratios and fractions often express comparative information in scaled-down form. For example, if a class consists of 15 boys and 10 girls, we might say that the ratio of boys to girls is 3 to 2, or that $\frac{3}{5}$ of the class is boys. We could also say that 60% of the class is boys, a kind of scaling up. *Stretching and Shrinking* lays a solid foundation of visual imagery to support the basic notion of scaling.

Research on students' understanding of proportional reasoning shows that moving from additive reasoning to multiplicative reasoning is difficult for students. Having experiences with geometric instances of proportional reasoning before concentrating on more numerical situations helps students in two ways: it gives students concrete experiences with visual representation of ratio comparisons, and it begins the work of helping students see the difference between reasoning by taking differences and reasoning by comparing ratios. This is why *Stretching and Shrinking* is in the CMP curriculum before *Comparing and Scaling*. The idea of ratio comparison was introduced there, along with informal ideas of equivalent ratios. These ideas are extended in the current unit. In *Moving Straight Ahead*, students will see proportional reasoning related to linear equations that pass through the origin.

In *Stretching and Shrinking*, the problem was finding dimensions of a larger (or smaller) physical or graphical model while preserving the relative size of the component parts so that the figures remained mathematically similar. The same ideas and ways of thinking developed in *Stretching and Shrinking* become powerful ways of thinking about ratios. The goal is the same in many ratio situations—to scale a pair of ratios up or down to determine whether they are equal.

A comparison problem in *Stretching and Shrinking* that called for finding the missing part of a ratio equivalent to a given ratio is the same as solving a proportion in *Comparing and Scaling*. For example, suppose you have a rectangle with dimensions of 5 cm by 7 cm. You want to draw a larger, similar, rectangle with the side corresponding to 5 cm being 15 cm. What would the other dimension be? This is an identical question: If roses are 5 for \$7, how much will 15 roses cost? In each case, we are dealing with the given ratio of 5 to 7 and looking for the equivalent ratio of 15 to x. *Stretching and Shrinking* precedes *Comparing and Scaling* to give students experience with these ideas in a more concrete geometric context.

To summarize, the broad purposes of this unit are twofold. First, to develop students' ability to make intelligent comparisons of quantitative information using ratios, fractions, decimals, rates, unit rates, and percents. Second, to use quantitative comparison information to make larger or smaller scale models or scale rates and ratios up and down. An additional goal of this unit is to have students not only learn different ways to reason in proportional situations, but to recognize when such reasoning is appropriate.

Many important mathematical applications involve comparing quantities of one kind or another. In some cases, the problem is simply deciding which of two quantities is greater and describing how much greater it is. In such instances, we subtract to find a difference. This is what students deal with in elementary school. In fact, comparison by addition or subtraction comes first in students' mathematics experiences. This way of thinking becomes inappropriately pervasive in any situation requiring comparison.

Summary of Investigations

Investigation 1

Making Comparisons

Investigation 1 focuses on the language of comparisons and ratios in the context of advertising. Some content connects to questions asked of students in the sixth-grade work on percents and data analysis. Students learn what different kinds of comparative statements mean about the data that is given. They are asked to write comparative statements that describe data. Questions are asked that engage students in making comparisons and checking the accuracy of statements given. The important question of how you decide whether to use a difference, ratio, fraction, or percent to make a particular comparison is raised.

Investigation 2

Comparing Ratios, Percents, and Fractions

Investigation 2 builds on the variety of strategies for making comparisons—ratios, percents, and fractions—that arose in Investigation 1. The intent is to see how information in each of these forms provides the information needed to derive either of the other forms. Students investigate in more depth how ratios can be formed and scaled up or down to find equivalent ratios. This investigation more directly raises issues with comparison by finding differences.

Investigation 3

Comparing and Scaling Rates

Investigation 3 takes a specific focus on rates, scaling rates, and finding and interpreting unit rates as strategies. The investigation looks at scaling in numerical contexts; the connection to such proportional reasoning problems in geometry is made in Investigation 4. Rate tables are introduced as a tool for using scaling rates as a strategy for comparison. Students are asked to draw rate tables. They are also asked to write rules or equations. The ideas of average speed and constant speed are used. Students explicitly learn to use unit rates and to write equations and rules based on unit rates. Problem 4 confronts students with the need to label rates and unit rates carefully. When you divide to find a unit rate, determining what the division gives you is essential to making the comparison. Here students look at the measurement labels for assistance in determining what the quotient means.

Investigation 4

Making Sense of Proportions

Investigation 4 helps students write and use proportions to solve problems and make comparisons. All of the problems in this investigation can be posed in classical $\frac{a}{b} = \frac{x}{d}$ or $\frac{a}{b} = \frac{c}{x}$ form; yet, they are solved in a variety of equivalent ways. It is important that students learn different ways to reason in proportional situations, and recognize *when* such reasoning is appropriate. The strategies used to solve problems are based on students' knowledge of equivalent fractions. In one case, a geometric context ties to earlier work. In Problem 4.3, we look more systematically for an efficient strategy for solving proportions. We do not, however, cover cross multiplication. We have made a commitment to help students make sense of the strategies they use and feel that efficiency is only effective if students truly make sense of what they are doing. Therefore, we focus on scaling ratios up and down as a way of solving proportions. This builds on the substantial foundation for understanding and using equivalent fractions in the sixth-grade curriculum.

Mathematics Background

The subtitle of *Comparing and Scaling* is *Ratio, Proportion, and Percent.* This subtitle makes clear that the heart of the unit goals is to recognize when making comparisons using these strategies is appropriate, then to use these strategies with understanding and efficiency.

Scaling Ratios as a Strategy

To compare two or more related measures or counts, such as 3 roses for $5 and 7 roses for $9, you need strategies that allow the related pairs of numbers to be compared. Simple subtraction will not tell you what you want to know. Enter the world of ratio and proportion. A proportion is a statement of equality between two ratios. In this example, you need to find a way to scale the ratios of 3 to 5 and 7 to 9 so that they can be directly compared. Many students think these two ratios are the same, reasoning that 4 has been added to each of the numbers 3 and 5 to get 7 and 9. This is an example of students' misconceptions about when additive comparisons are appropriate. If you appropriately scale both ratios so that either the number of roses or the costs are the same, you are then left with a simple comparison of the quantities that are not the same. The two possibilities are shown below.

If you want to scale the costs to be the same, the kind of thinking is the same as that for finding a common denominator: look for a number that represents a multiple of the two numbers 5 and 9. If you scale to make the prices the same (that is, $45), then the answer is immediately obvious.

$$\frac{3}{5} = \frac{3 \times 9}{5 \times 9} = \frac{27}{45} \text{ and } \frac{7}{9} = \frac{7 \times 5}{9 \times 5} = \frac{35}{45}$$

You can now compare the ratios 27 roses for $45 and 35 roses for $45. Clearly the second option gives more for the same amount of money.

Let's scale the numerators to be the same.

$$\frac{3}{5} = \frac{3 \times 7}{5 \times 7} = \frac{21}{35} \text{ and } \frac{7}{9} = \frac{7 \times 3}{9 \times 3} = \frac{21}{27}$$

You can now compare the ratios 21 roses for $35 and 21 roses for $27. Again the best buy is obvious.

This example underscores the relationship between the mathematical thinking used to find common denominators or common numerators in work with equivalent fractions and that was used to find equivalent ratios. Ratios are written in several forms. Some of the most often used are 2 to 3, or 2 : 3, or $\frac{2}{3}$. In the example, the convenience of writing the ratios as fractions helps the thinking needed for scaling the ratios up. However, we must make sure that students can differentiate between a ratio written as a fraction and a fraction representing the fractional part of a whole. We address this in the next section.

Using Ratio Statements to Find Fraction Statements of Comparison

The statement "the ratio of girls to boys in a class is 15 girls to 9 boys" can be written as the fraction $\frac{15}{9}$, but it does not mean that the fraction of students in the class that are girls is $\frac{15}{9}$. This is confusing for students and leads some teachers to avoid the fraction form for writing a ratio. We have chosen to confront the confusion by asking the fraction question directly.

> Maria says the fraction of the class that is girls is $\frac{15}{9}$. Bob says the fraction of the class that is girls is $\frac{15}{24}$. Who is correct and why?

The correct answer hinges on recognizing that a new quantity is actually used to find the fraction of students in the class that are girls. The total number of students in the class is needed. This is the sum of the numbers of boys and girls, 24. The part to whole comparison is $\frac{15}{24}$, and Bob is correct. Now we turn to another strategy for solving the roses problem.

Per Quantities: Finding and Using Rates and Unit Rates

If you compute the price per rose, you will have a rate comparison for the roses problem. In the 3 for $5 deal, the unit rate is $1.67. The price per rose in the 7 for $9 deal is $1.29—clearly the better price. Alternatively, at the 3 for $5 price, 7 roses would cost $11.67. This is a different comparison with the same result. Let's explore this strategy a bit further.

Here are two ratios that suggest rates:

> Sascha goes 5 miles in 20 minutes on the first part of his bike ride. On the second part, he goes 8 miles in 24 minutes. On which part is he riding faster?

Many students will intuitively want to divide the miles number and the minutes number to get a result, but they sometimes lose track of which one is divided into the other. Consequently they produce a number, but have no idea what the number means in the problem. Here the comparison can be made in two different ways by computing different unit rates. Let's look at each.

Suppose a student decided to divide 5 by 20 and 8 by 24. She gets the two numbers 0.25 and 0.333. What do these numbers mean? She might have divided 20 by 5 and 24 by 8. This division gives 4 and 3. What do these numbers mean? You have to know before you can decide what they tell us about the two legs of the bike ride. So start again and this time carry the label with the quantities.

$\frac{5 \text{ miles}}{20 \text{ minutes}} = 0.25$ *miles per minute* and

$\frac{8 \text{ miles}}{24 \text{ minutes}} = 0.333\ldots$ *miles per minute*

Now the comparison is clear. The times are the same, 1 minute, and the distances can be directly compared. 8 miles in 24 minutes is faster.

But, you could divide the other way:

$\frac{20 \text{ minutes}}{5 \text{ miles}} = 4$ *minutes per mile* and

$\frac{24 \text{ minutes}}{8 \text{ miles}} = 3$ *minutes per mile*

Now you see that the lesser number tells the correct answer, 8 miles in 24 minutes.

What makes unit rates so interesting, and somewhat difficult, for students is that you have two options when you divide two numbers. The units help students think through such situations with the goal of building the flexibility to use either set of unit rates to compare the quantities.

One of the recurring themes of these materials is that we can represent data in different ways and that each way may tell us something that is not as obvious from other representations. The comparison in the rose example can be made in several ways: for example, using unit rates, comparing the ratios in fraction form to determine which is greater, or scaling both rates until the price is the same or the number of roses is the same. Developing strategies for deciding what the comparison situation calls for and for making comparisons are major goals of this unit.

Relating Ratios, Fractions and Percents

It is often desirable to change one form of comparison statement to another. The question is, can you write a percent statement given either a ratio or a fraction statement, and can you write a ratio or fraction statement given a percent comparison statement? Let's explore this with an example.

The ratio of concentrate to water in a mix for lemonade is 3 cups concentrate to 16 cups water. The questions you might ask are: "What fraction of the lemonade will be concentrate?" or "What percent of the lemonade will be concentrate?"

First find the total cups the recipe makes. It makes 19 cups. Then write the fraction of the lemonade that is concentrate, $\frac{3}{19}$. Now finding the percent is easy. Just divide the concentrate by the total, $3 \div 19 = 0.15789\ldots$ or about 15.8% concentrate.

Suppose you know that the percent of boys in a class is 48% and you want to write this as a ratio. You can think of the percent as a scaling of the ratio representing boys and girls up to a total of 100. So the girls are 52% of the class and the ratio of boys to girls is 48 to 52. You can scale this ratio down to 12 boys to 13 girls. The powerful thing about these related representations is the flexibility it gives us to choose the form of representation that describes the situation best for our purposes.

One caution about such changes of representation is that the choice to make these changes of form should be judged against whether the computations you do will have meaning. For example, in many rate situations, such as miles per gallon, trying to compute a percent does not make sense because the addition to get a total does not make sense. Miles covered plus gallons of gas used is a meaningless total. When the ratio can be thought of as part of a whole, the change of form we described makes sense (for example, white paint to blue paint in a mix, or high-fiber to high-protein nuggets in food for a baby chimp).

Proportions and Proportional Reasoning

The related concepts and skills in this unit are often referred to as *proportional reasoning*. Forming ratios in order to make comparisons is the heart of proportional reasoning. What is a proportion? A proportion is simply a statement of equality between two ratios. What makes this idea powerful is that if we know a ratio is equivalent to another, but we do not know both terms of one of the ratios, we can use what we already know about scaling or finding equivalent fractions to find the missing part of a proportion. Again, let's look at an example.

> It takes Glenda 70 steps on the elliptical machine to go 0.1 mile. When her workout is done, she has gone 3 miles. How many steps has she taken on the machine?

Here is a proportion and a solution for the number of steps that Glenda made.

$$\frac{70 \text{ steps}}{0.1 \text{ miles}} = \frac{x \text{ steps}}{3 \text{ miles}} = \frac{70 \times 30 \text{ steps}}{0.1 \times 30 \text{ miles}} = \frac{2{,}100 \text{ steps}}{3 \text{ miles}}$$

The first ratio in the proportion is scaled up by multiplying both the numerator and the denominator by 30. Thus, the denominator equals the denominator of the ratio with the unknown, x. This allows us to read the value of x directly since we know that if the two fractions are equivalent and have the same denominator, the numerators are also the same. The strategy we use to find the number by which we multiply, or "scale," is the same as the thinking process we use to find common denominators for fractions.

How far you go in formalizing the solving of proportions will depend on you and your students. We highly recommend that you do not impose solution strategies that have no meaning for the students. While cross multiplication is efficient, for most students at this level it is used without any understanding of why it works and consequently is often misused. We believe that students are better served by having the time to learn to think through situations requiring solving proportions and develop flexibility in approaching a problem so that easy possible solution strategies are not missed in a rush to an algorithm. This approach also builds on mathematics students already know and ways of thinking that they have already acquired. Helping students want to make sense of mathematics is encouraging a kind of thinking and flexibility that will allow them to feel confident to tackle problems that do not look exactly like ones they have already solved. Part of the goal of this unit is for students to learn to make judgments about the situation and to choose methods for comparing and for scaling.

Cross-Multiplying

If someone mentions cross-multiplication and the students seem interested, don't just give a procedure for cross-multiplication. Develop the idea based on what your students already know—finding common denominators. (If we use the product of the original denominators as a common denominator, the numerators will be cross products). In Question C part (2) of Problem 4.1, $\frac{7}{12} = \frac{x}{9}$, the common denominator will be 108. In the first fraction we have to multiply the numerator and denominator by 9. In the second fraction we need to multiply the numerator and the denominator by 12 to make the denominators the same.

Because the denominators are now the same, we need to find the value of x that makes the numerators equal. So we have to find x when $12x = 63$. This means that x must be 5.25. $\frac{7}{12} = \frac{x}{9}$ is equivalent to $\frac{63}{108} = \frac{12x}{108}$ is equivalent to $63 = 12x$, therefore $x = 5.25$.

So in a sense, cross-multiplying asks the question: What would be the numerator if these two fractions had a particular common denominator (the product of the original denominators)?

Helping students to make their own reasoning explicit can lead to a generalized method of solving proportions. For example, when many students solve this proportion $\frac{3}{7} = \frac{x}{343}$, they do the following arithmetic.

$$(343 \div 7) \times 3 = \frac{343}{7} \times 3 = \frac{343 \times 3}{7}$$

The division $343 \div 7$ finds the scaling factor by which we need to scale the 3.

Consequently, for solving a general proportion $\frac{a}{b} = \frac{x}{c}$, we can follow the same reasoning: find the scaling factor by computing $c \div b$, then multiply the scaling factor by a. So the arithmetic we actually perform is scale factor $\times$ the known numerator. In symbols this is $\frac{c}{b} \times a = x$ or $\frac{c \times a}{b} = x$.

With the unknown in the denominator, find the scale factor using the numerators so that we can scale the denominators to find the unknown. To solve $\frac{a}{b} = \frac{c}{x}$, we first find the scale factor by which we can make the numerators the same, $c \div a$. Then we have to scale the denominator to see what is equal to x. This gives $\frac{c}{a} \times b = x$ or $\frac{c \times b}{a} = x$.

An alternative strategy can be built using fact family ideas. To solve $\frac{a}{b} = \frac{c}{x}$, think of the equation as $\frac{a}{b} = c \div x$. From fact families, we can say that $x = c \div \frac{a}{b}$. Rewriting the right side with common denominators gives $x = \frac{cb}{b} \div \frac{a}{b} = cb \div a$, or $x = \frac{cb}{a}$.

These are the equations we would get by cross-multiplication, but here the explanation is built on students' ways of reasoning.