Unit Introduction

What Do You Expect?

Probability and Expected Value

Goals of the Unit

- Interpret experimental and theoretical probabilities and the relationship between them
- Distinguish between equally likely and non-equally likely outcomes
- Review strategies for identifying possible outcomes and analyzing probabilities, such as using lists or tree diagrams
- Determine if a game is fair or unfair
- Analyze situations that involve two stages (or two actions)
- Use area models to analyze situations that involve two stages
- Determine the expected value of a probability situation
- Analyze situations that involve binomial outcomes
- Use probability and expected value to make decision

Developing Students' Mathematical Habits

The overall goal of *Connected Mathematics* is to help students develop sound mathematical habits. Through their work in this unit, students learn important questions to ask themselves, such as:

- *What are the possible outcomes for the event(s) in this situation?*
- *Are these outcomes equally likely?*
- *Is this a fair or unfair situation?*
- *Can I compute the theoretical probabilities or do I conduct an experiment?*
- *How can I determine the probability of the outcome of one event followed by a second event?*
- *How can I use expected value to help me make decisions?*

Mathematics of the Unit

Pearson Prentice Hall
Professional Development

Overview

What Do You Expect? is the second probability unit in the *Connected Mathematics* curriculum. The work in this unit assumes that students are familiar with the basic ideas of probability that are presented in the grade 6 unit, *How Likely Is It?* If some or all of your students have not explored the concepts covered in that unit, you will need to prepare them for the mathematics they will encounter in *What Do You Expect?* or consider teaching *How Likely Is It?* If your students have studied *How Likely Is It?,* Investigation 1 of this unit should be a sufficient review, as well as an extension, of the ideas with which they are already acquainted. Through their work in this unit, students will deepen and expand their understanding of basic probability concepts.

Summary of Investigations

Investigation 1

Evaluating Games of Chance

Investigation 1 uses a variety of situations that provide students a chance to review both experimental and theoretical probabilities, equally likely events, fair/unfair games, and strategies for determining theoretical probabilities. Spinners, choosing marbles from two buckets, and rolling two number cubes provide the settings. These situations also introduce two-stage events. For example, students spin a spinner twice and then look at the outcomes of a match/no-match.

Investigation 2

Analyzing Situations Using an Area Model

Investigation 2 uses the area model as a way to analyze the theoretical probability of two-stage events. The two-stage events used are spinning two spinners, choosing paths in a game, and choosing a marble at random from a container chosen at random.

Investigation 3

Expected Value

In Investigation 3, the two-stage event is a one-and-one free-throw situation. A player with a 60% free-throw shooting average goes for a one-and-one. That is, the player shoots the first free throw and then either takes a second free throw (if the first one was made) or does not get a second chance (if the first free throw was missed). After determining experimental probabilities that the player will get a score of 0, 1, or 2, students find the theoretical probability by using an area model. Students determine the long-term average (expected value) for the situation and explore expected value in a variety of different probability settings.

Investigation 4

Binomial Outcomes

Students are introduced to binomial situations by taking a four-item true-false quiz where each answer is determined by tossing a coin. Students then find the expected value (or average score) for guessing. Students also use lists or trees to determine outcomes. The situations lead naturally to Pascal's Triangle, which is explored in the ACE.

Mathematics Background

The following is a summary of the basic ideas that are covered in the grade 6 probability unit, *How Likely Is It?*, and descriptions of the new mathematical ideas students will encounter in *What Do You Expect?*

Basic Probability Concepts

The term *probability* is applied to situations that have uncertain outcomes on individual trials but a predictable pattern of outcomes over many trials. For example, when we toss a fair coin, we are uncertain whether it will come up heads or tails; but we do know that, over the long run, we will get heads about half of the time and tails about half of the time. This does not mean that we can't get several heads in a row. Nor does it mean that if we get a head on one toss, we are more likely to get tails on the next. This concept of uncertainty on an individual outcome but predictable regularity in the long run is often difficult for students. Students need a variety of experiences that challenge their prior conceptions before they grasp this basic concept of probability.

If we toss a tack into the air, we know that it will land either on its head or its side. If we toss a tack many times, we can use the ratio of the number of times it lands on its side to the total number of tosses to estimate the likelihood that the tack will land on its side. Since this ratio is found by experimentation, it is called an *experimental probability*. Many uses of probability in daily life, such as weather forecasts and sports predictions, are based on experimental probabilities.

This unit offers many opportunities for students to collect data through experimentation and to use their data to assign experimental probabilities to the possible outcomes. It is important for students to realize that comparison of samples with small numbers of trials may show wide variation among the samples, and that only through experimentation over many trials can good estimates be made about what will happen in the long run. In other words, experimental probabilities must be based on a great number of trials relative to the number of possible outcomes in order to have reasonable predictability. In some situations, such as tossing a fair coin, we can also find a *theoretical probability*. We know that a fair coin will land either heads up or tails up and that each outcome is *equally likely*. Since each of the two outcomes is equally likely, the probability that a fair coin will land heads up is 1 out of 2, or $\frac{1}{2}$. In a situation where all events are equally likely, the theoretical probability can be expressed as:

$$P\text{ (outcome)} = \frac{\text{number of possible favorable outcomes}}{\text{total number of possible outcomes}}$$

The theoretical probability of getting a head on one toss of a fair coin is:

$$P(\text{head}) = \frac{\text{number of possible favorable outcomes}}{\text{total number of possible outcomes}} = \frac{1}{2}$$

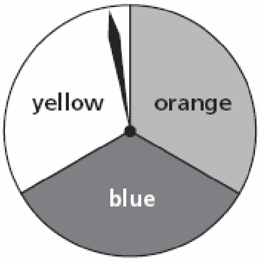
Another example of a situation for which we can find a theoretical probability is the rolling of a number cube. The six possible outcomes are 1, 2, 3, 4, 5, and 6 and each are equally likely to occur on any single roll. Thus, $P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$. We can use this theoretical probability to estimate that if a number cube is rolled many times, we could expect each number to be rolled about $\frac{1}{6}$ of the time.

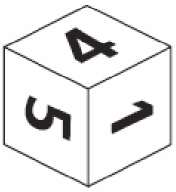
Probabilities, whether obtained through theoretical analysis or experimentation, are useful for predicting what should happen over the long run. Yet, a probability does not tell us exactly what will happen. If we toss a coin 40 times, we may not get exactly 20 heads; but if we toss a coin 1,000 times, the ratio of heads to the number of tosses is likely to be fairly close to $\frac{1}{2}$. Experimental data gathered over many trials should produce probabilities that are close to the theoretical probabilities; this idea is sometimes called the *Law of Large Numbers* (see discussion of this on page 9). If we can calculate a theoretical probability, we can use it to predict what will happen in the long run rather than having to rely on experimentation alone.

Theoretical Probability Models: Lists and Tree Diagrams

Students who have studied the grade 6 probability unit, *How Likely Is It?,* have already learned quite a bit about conducting simulations to find experimental probabilities and making organized lists of possible outcomes or tree diagrams to find theoretical probabilities. In this unit, they will continue to work with these familiar strategies, while learning a new strategy for finding theoretical probabilities for two-stage events—constructing area models to represent the possible outcomes.

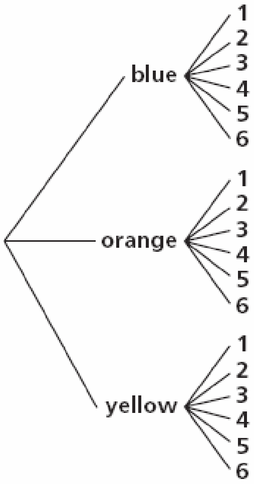
Tree diagrams can be used throughout the unit. They offer students a way to determine all the possible outcomes in a situation systematically, particularly those that are two-stage situations. For example, suppose a spinner divided into three equal sections is spun (stage 1) and a six-sided number cube is rolled (stage 2).





The possible outcomes can be shown in a list and a tree diagram.

Spinner	Number Cube
blue	1
blue	2
blue	3
blue	4
blue	5
blue	6
orange	1
orange	2
orange	3
orange	4
orange	5
orange	6
yellow	1
yellow	2
yellow	3
yellow	4
yellow	5
yellow	6



In this unit, students use tree diagrams to find the number of equally likely outcomes in situations with a great number of possible outcomes. Tree diagrams are particularly useful for listing outcomes in situations involving a series of actions in which each outcome of a particular action is equally likely. Such situations include rolling a number cube twice, rolling two number cubes, tossing a coin four times, tossing four coins; or choosing several items from a menu, such as a sandwich, a drink, and a dessert. However, when there are many possibilities at a particular stage, tree diagrams can become unwieldy.

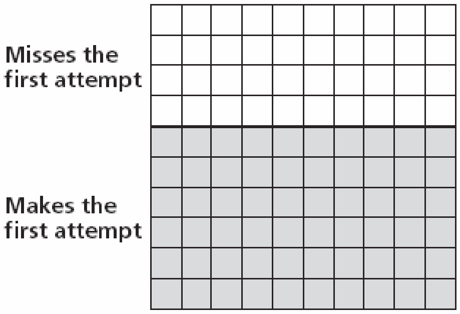
Tree diagrams can be used as a basis for understanding the multiplication of probabilities. Multiplication occasions do arise, but building facility with determining such situations is beyond the scope of this unit. Students do not yet understand enough about probability to know when and why it is appropriate to multiply probabilities.

Theoretical Probability Models: Area Models

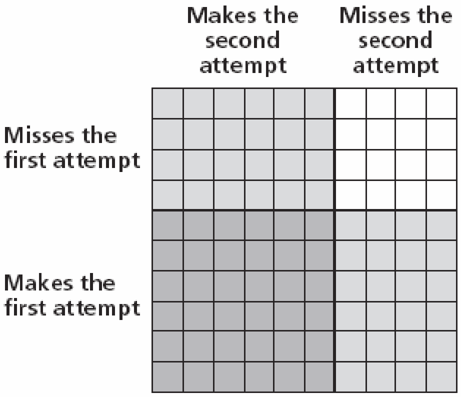
Area models, like tree diagrams, are useful for finding probabilities in situations involving successive events, such as a basketball player who is allowed to attempt a second free throw only if the first succeeds. Unlike tree diagrams, an area model is particularly powerful in situations in which the possible outcomes are not equally likely.

The following steps demonstrate how to create an area model to show the probability that Nishi, a player with a 60% free-throw average, will score 0, 1, or 2 points in a two-try free-throw situation in basketball. In a two-try situation, the player will get to attempt a second free throw whether or not the first free throw succeeds.

The first try has two possible outcomes, making or missing the basket. The probability of missing the basket is 40% or 0.4 or $\frac{40}{100}$. The probability of making the basket is 60% or 0.6 or $\frac{60}{100}$. The grid below is shaded to indicate this.

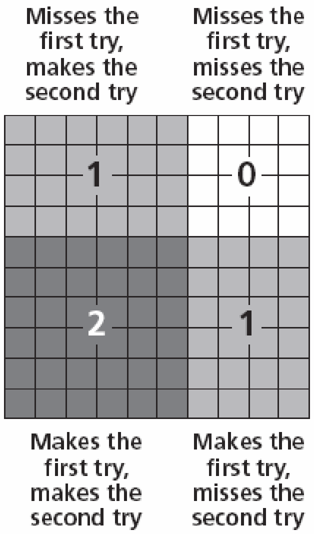


The second try has the same two possible outcomes. These are marked vertically on the grid.

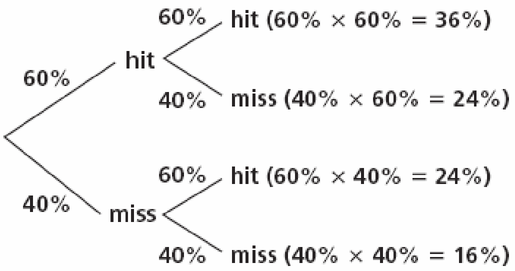


The probability that Nishi will make her second try is 60% of the time that she has already made her first try, or 36% of the time. The probability that Nishi will miss her second try is 40% of the time that she makes her first try, or 24% of the time.

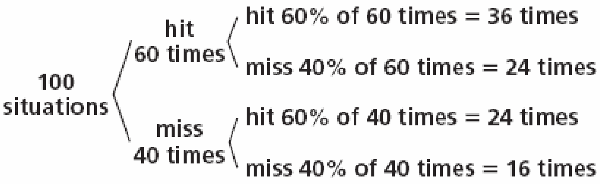
If she missed the first try, she still hits the second one 60% of the time. So the probability of getting a score of 0 is $\frac{16}{100}$, getting a score of 1 is $\frac{48}{100}$, and getting a score of 2 is $\frac{36}{100}$. The grid below indicates this, and each region is labeled with the number of points it represents.



To use a tree-diagram approach in a situation where outcomes are not equally likely, each branch of the tree must be weighted by the probability that it will be chosen. This idea is quite difficult for students at this stage to understand; they have used tree diagrams only in situations involving equally likely outcomes. An area analysis makes the weighting more obvious. It is not recommended that you introduce this idea to your students now, but shown here is a tree diagram that works.

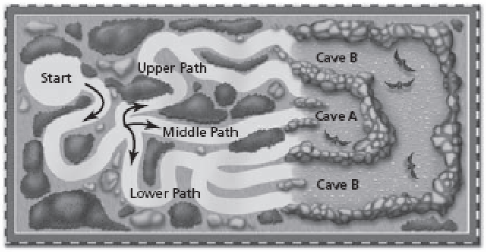


Students, however, will sometimes make a modified version of a weighted tree diagram, pictured below. Such a student might choose a large number of situations (here: 100), then indicate how many of these he would expect to occur on each first branch (here: 60 and 40, corresponding to Nishi's percent of free-throw success). Then each of these numbers is broken down proportionately for the next stage. In effect, this is the same idea as above, but is more accessible to students at this stage of their study of probability.

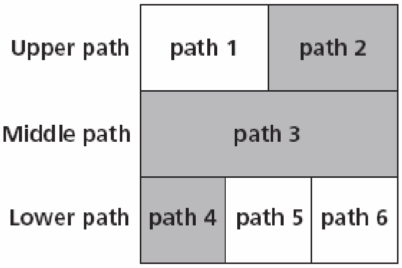


In Problem 3.1, students explore the probabilities of getting a score of 0, 1, or 2 for a person with a 60% free-throw average in a one-and-one situation.

Consider one more example of these ideas. In Investigation 2, students consider a path game (below) in which a player chooses a path at random at each intersection. Students are to figure out the probability of landing in either Cave A or Cave B. Note: The diagram and the analysis here show a different way of labeling the analysis than in the Investigation. This gives you an alternative strategy in case your students are struggling.

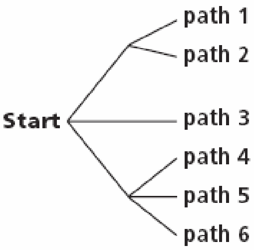


The area model for this game is first split into thirds to indicate the three equally likely paths at the first intersection: the upper path, the middle path, and the lower path. Then each of these thirds is split according to the later intersections (if any), resulting in the model on the following page.

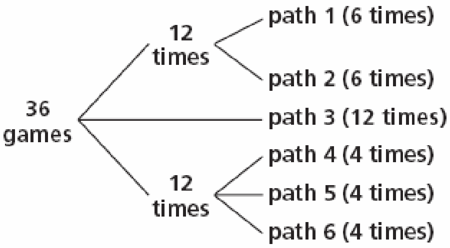


From the area model, it is clear that the 6 paths are not equally likely. Path 3, for instance, has a probability of $\frac{1}{3}$, while path 4 has probability of $\frac{1}{9}$.

A simple tree diagram would not show this:



But the modified tree diagram described below would in fact represent the differences in the probabilities for each path. Path 3 occurs 12 out of the 36 games, more than any of the other paths.



Compound Events and Multi-Stage Events

If you are interested in the probability of an event, A, happening, and there are several ways that A can happen, then A is a *compound event*. The probability of A happening is the sum of the probabilities of each possible way that A can happen. For example, if you toss two coins and are interested in finding the probability that you will get a match, there are two ways that A can happen. You can get two heads or two tails. The probability of A, $P(\mathrm{A})$, is the sum of the probabilities of each outcome where two coins match. $P(\mathrm{A}) = P(t, t) + P(h, h) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

An event is a *multi-stage event* if it takes more than one action to create an outcome. In the example above, event A is a two-stage event, since it takes the toss of two coins (or one coin, twice) to get an outcome. The possibilities are (t, t), (t, h), (h, t), and (h, h). The question is, what is the probability of each of these outcomes? If the coin is fair, then each coin toss has a probability of landing tails or heads. The coin tosses are *independent* of each other. How the coin lands on a given toss is not affected by any previous toss. Here $P(t, t) = P(t) \times P(t)$ or $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$. The same is true for each of the four possible outcomes.

For a player with a 60% free-throw average in a one-and-one situation, whether or not the player gets to take a second try depends on the result of the first try. Here the second try is *dependent* on the result of the first attempt. Thus, $P(0 \text{ points})$ can only be achieved in one way and that is to miss the first try. The probability of a miss on the first try is 0.4. Thus, $P(0 \text{ points}) = 0.4$. There are two possible outcomes resulting from a hit on the first try; the player can hit or miss the second try. So we have $P(1 \text{ point}) = P(h, h) = 0.6 \times 0.6 = 0.36$ and $P(h, m) = P(1 \text{ point}) = 0.6 \times 0.4 = 0.24$. In a one-and-one free-throw situation, we have three possible outcomes, one of which is a one-stage event (0 points) and two of which are two-stage events (1 point or 2 points). The deciding factor is that the second action is dependent on the result of the first action. The sum of all possible outcomes is $P(0 \text{ points}) + P(1 \text{ point}) + P(2 \text{ points}) = 0.4 + 0.24 + 0.36 = 1$.

Expected Value

The "in the long run" perspective of probability is key to understanding probability. Rather than *guarantee* what will happen on a particular trial or even in the short run, probability models *predict* what will happen in the long run over many trials. Often, this is the most valuable information we can gain about a probability situation: a prediction of the expected value of the situation. The *expected value* is the average of the payoff of each outcome weighted by its probability. It predicts long-run expectations.

In this unit, students are introduced to expected value in an informal yet concrete way. We do not expect them to develop a formal definition of expected value or to use a formula for finding it. In fact, students might never use the term expected value in their work in this unit, instead thinking of the concept as "what is expected in the long run." However, expected value is vocabulary that the student text uses frequently once it is introduced.

Expected value goes beyond basic probabilities. It uses value, such as points earned in a game or money won in a contest, to weight each possible outcome by then computing the average points or dollars we can expect per game or contest in the long run. You can think of expected value as a "weighted average."

Example 1

Consider the long-term average or expected value for a player with a 60% free-throw average in two-try free-throw situations.

If the player goes to the line 100 times, then he/she expects:

A score of 0 to occur 16 times for a total of 0 points and a score of 1 to occur 48 times for a total of 48 points and a score of 2 to occur 36 times for a total of 72 points.

The total number of points expected in 100 situations is $0 + 48 + 72 = 120$ points. The average number of points expected in 100 trials is 120 points $\div$ 100 trials or 1.2 points per trial.

Example 2

We could also arrive at this result by the computation

$\frac{16}{100}(0) + \frac{48}{100}(1) + \frac{36}{100}(2) = \frac{0}{100} + \frac{48}{100} + \frac{72}{100} = 1.2$, which shows each payoff weighted by the probability that it will occur.

The second example is closer to the mathematical definition of expected value but more conceptually difficult for students and is not directly addressed in this unit. Rather, students compute the expected value in steps, as shown in example 1.

Example 3

The expected value for a player with a 60% free-throw average in a one-and-one situation is:

$$\frac{40}{100}(0) + \frac{24}{100}(1) + \frac{36}{100}(2) = \frac{96}{100} = 0.96$$

A natural question is what free-throw average for a player gives an expected value of exactly 1 point. This question has a surprising answer. The following is a mathematical analysis that shows how problems such as this one may be revisited in high school when students are ready to solve quadratic equations.

Let p represent the probability that a player will make the free throw. Then, $(1 - p)$ represents the probability that the player will miss the free throw. Thus, the probability of

- making 0 points is $(1 - p)$
- making 1 point is $p(1 - p)$
- making 2 points is $p \times p$, or p^2

The expected value is thus:

P(2 points) × 2 + P(1 point) × 1 + P(0 points) × 0

Symbolically,

$$p^2 \times 2 + p(1 - p) \times 1 + (1 - p) \times 0$$

Setting the expected value equal to 1, we can solve for p:

$$p^2 \times 2 + p(1 - p) \times 1 + (1 - p) \times 0 = 1$$
$$2p^2 + p - p^2 + 0 = 1$$
$$p^2 + p = 1$$

Using the quadratic formula to solve the resulting quadratic equation, $p^2 + p - 1 = 0$, yields:

$$p = \frac{-1 + \sqrt{1 + 4}}{2} = \frac{-1 + \sqrt{5}}{2}$$

This is the golden ratio, which is approximately 0.6180339887. The golden ratio is the proportion of length to width of a rectangle that many people consider to be the most beautiful rectangle. Many ancient Greek buildings were built with facades that incorporate this ratio.

More on Independent and Dependent Events

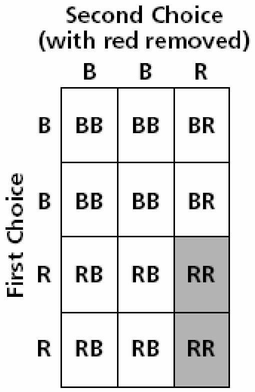
Please note the terms *independent* and *dependent events* are not mentioned in this unit. Naming these ideas can wait until a later course in probability. In this unit, students need only to make sense of each situation and apply the appropriate probability at each stage.

The idea of *independent* and *dependent* events is introduced informally. A more formal approach is often a major focus of probability study in high school and college courses. Yet, we feel it is important to introduce this concept because many students working through a basic probability unit such as this one develop the belief that *all* events are independent.

Suppose you twice choose a marble from a bag containing two red marbles and two blue marbles. If you replace the chosen marble after the first choice, the two choices will be independent of each other, because what you choose the first time will not affect what you choose the second time. If you do not replace the chosen marble, the second choice will be dependent on the first choice, because the probability of choosing each color the second time depends on the color chosen on the first choice. For example, if you choose a red marble the first time and do not replace it, the probability of chosing a red marble the second time is $\frac{1}{3}$ rather than $\frac{1}{2}$. Yet if you had chosen a blue marble the first time, the probability of choosing red the second time would be $\frac{2}{3}$. It is in this sense that the probability of choosing a red on the second choice is a dependent probability.

In this unit, students analyze dependent events by using the situation to help make sense of the sequence of actions. They look at the context and determine the sequence of actions and the possibilities at each step in the sequence. The steps in the sequence guide the apportioning of the total area in an area model, or the designing of a tree diagram representing all possible outcomes. Then, each portion of area in an area model, or each path on a tree diagram, is compared to the total area or the total number of possible outcomes to form probability statements.

Consider an area model for the marbles without replacement:



Second Choice (with red removed)

First Choice	B	B	R
B	BB	BB	BR
B	BB	BB	BR
R	RB	RB	▨ RR
R	RB	RB	▨ RR

The probability of choosing two reds is $\frac{1}{6}$. Note that the probability of choosing a red on the second choice is greater if blue was chosen on the first choice.

As students use an area model to make sense of two-stage probability situations, take any opportunity to help those who seem ready to see the connection to multiplying probabilities. For example, in the preceding 60% two free-throw situation,

$$P(\text{score of } 0) = \frac{40}{100} \times \frac{40}{100} = \frac{16}{100}$$

$$P(\text{score of } 1) = \frac{60}{100} \times \frac{40}{100} + \frac{40}{100} \times \frac{60}{100} = \frac{48}{100}$$

$$P(\text{score of } 2) = \frac{60}{100} \times \frac{60}{100} = \frac{36}{100}$$

As an area model is also used to develop an understanding of the multiplication of fractions, many students will see this connection naturally.

The Law of Large Numbers

The Law of Large Numbers tells us that as we conduct more and more trials, the probabilities drawn from the experimental data should grow closer to the actual probabilities. This idea is difficult for students to grasp; they need time to experiment to develop an understanding of this concept. As you work with the class, talk about the need for many trials in conducting an experiment to find experimental probabilities.

Binomial Events and Pascal's Triangle

Many interesting probability situations are of the type where there are exactly two equally likely possible outcomes: yes or no, boy or girl, true or false, heads or tails, etc. These are called binomial events. If students guess at every answer for a five-item true/false quiz, there are 32 ways to answer the quiz, but only one of them has all five answers correct. The probability of getting all five answers correct is $\frac{1}{32}$. A similar situation involves the families in the town of Ortonville. Each family has exactly five children and they all agree to name their children the same names. There are 32 ways to arrange five children according to numbers of boys and girls (BBGGG, BGBGG, GGGGG, etc.) The probability of a family having exactly five girls is $\frac{1}{32}$.

The probability of having two boys and three girls is $\frac{10}{32}$. Once one binomial situation has been analyzed it is easy to analyze another binomial situation.

Pascal's Triangle is used to analyze binomial probabilities. The triangle of numbers is named after the seventeenth century mathematician Blaise Pascal. However, the array was in existence long before this. The first five rows are below:

Pascal's Triangle

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

Pascal's Triangle and a Coin Toss

Row	Number of Outcomes
1	Tossing 1 coin
2	Tossing 2 coins
3	Tossing 3 coins
4	Tossing 4 coins
5	Tossing 5 coins

Pascal's Triangle and a True/False Test

Row	Number of Outcomes
1	True/false test with 1 question
2	True/false test with 2 questions
3	True/false test with 3 questions
4	True/false test with 4 questions
5	True/false test with 5 questions

The first row states that there are two possible outcomes for tossing a coin, a head or a tail, and there are two possible outcomes for answering a true/false question, true or false. The fifth row states that there is 1 way to get five heads, (1 way to answer all questions true), 5 ways to get four heads and one tail (5 ways to answer four questions true and one question false), 10 ways to get three heads and two tails (10 ways to answer three questions true and two question false), 10 ways to get two heads and three tails (10 ways to answer two questions true and three question false), 5 ways to get one head and four tails (5 ways to answer one questions true and four question false), and 1 way to get five tails (1 way to answer all questions false). A similar analysis can be used for any other binomial situation.

Pascal's Triangle is only presented in an ACE, but students recognize the similarity between the binomial situations and can use previous results to analyze a new situation. An example is a problem that involves a Baseball Series between the evenly matched Gazelles (G) and Bobcats (B). The Gazelles have won the first two games. What is the probability that the series will end in four games? Five games? Six games? Seven games? To answer these questions students analyze the possible outcomes of the last five games. Again there are 32 outcomes. The probability of ending in 4, 5, 6, or 7 games equals ($\frac{1}{4}$). However, the Gazelles have a greater chance of winning the series.