Unit Introduction

Thinking With Mathematical Models

Linear and Inverse Variation

Goals of the Unit

- Recognize linear and nonlinear patterns from verbal descriptions, tables, and graphs and describe those patterns using words and equations
- Write equations to express linear patterns appearing in tables, graphs, and verbal contexts
- Write a linear equation when given specific information, such as two points or a point and the slope
- Approximate linear data patterns with graph and equation models
- Solve linear equations
- Develop an informal understanding of inequalities
- Write equations describing inverse variation
- Use linear and inverse variation equations to solve problems and to make predictions and decisions

Developing Students' Mathematical Habits

The overall goal of *Connected Mathematics* is to help students develop sound mathematical habits. Through their work in this and other algebra units, students learn important questions to ask themselves about any situation that can be represented and modeled mathematically, such as:

- *What are the key variables in this situation?*
- *What is the pattern relating the variables?*
- *What kind of equation will express the relationship?*
- *How can I use the equation to answer questions about the relationship?*

Mathematics of the Unit

Pearson Prentice Hall Professional Development

Overview

Functions and the equations that represent them are invaluable tools for quantitative reasoning throughout the physical, biological, social, and management sciences. Development of student understanding of, and skill with, functions and algebraic equations began with *Variables and Patterns* and *Moving Straight Ahead* in grade 7. This first grade 8 unit reviews linear functions and equations and introduces concepts associated with nonlinear functions—in particular, inverse functions—that will be addressed in subsequent algebra units.

In applying algebra to problem-solving tasks, a critical step is representing relationships in symbolic form so that the tools of algebra can be applied effectively. In some situations, the stated problem conditions can be used to write algebraic equations for functions directly and precisely. In other cases, relationships between key variables are only suggested by data patterns. Such relationships can be approximated by mathematical functions, but cannot be precisely described by them. When algebraic equations are used to represent patterns in data from experiments or surveys, the resulting functions are called *mathematical models* of the underlying relationships. The models can be used to make calculations and to estimate answers to questions about the relationships. One of the central goals of *Thinking With Mathematical Models* is to develop student understanding of, and skill with, elementary aspects of the modeling process.

Two of the simplest and most common types of relationships are *direct variation* and *inverse variation*. Simple direct variation models are those that can be expressed with equations in the form $y = kx$. CMP students are familiar with direct variation (although not by name) as a special case of linear relationships. Specifically, direct variations are linear relationships with y-intercepts of 0. Inverse variation models are those that can be expressed with equations in the form $y = \frac{k}{x}$. This unit introduces inverse variation and helps students develop facility in working with this type of relationship in several common contexts.

Summary of Investigations

Investigation 1

Exploring Data Patterns

The central objectives of this investigation are to refresh student understanding of linear relationships and to contrast linear and nonlinear patterns.

Although students work with inverse and quadratic relationships in this investigation, they are not expected to name these specific types of relationships or to represent them symbolically. Students are formally introduced to inverse relationships in Investigation 3 of this unit and to quadratic relationships in the *Frogs, Fleas, and Painted Cubes* unit.

Investigation 2

Linear Models and Equations

This investigation introduces the idea of using a mathematical model to approximate patterns in data. Students review methods of writing linear equations to match given information and methods for solving linear equations. They also use informal methods to solve inequalities.

Investigation 3

Inverse Variation

The overall goal of Investigation 3 is to acquaint students with inverse variation, one of the fundamental nonlinear patterns of variation. Students should become comfortable with interpreting numeric and graphic patterns associated with inverse variations and representing such relationships with symbolic equations.

Mathematics Background

There are three central mathematical ideas developed in this unit: linear functions, equations, and inequalities; inverse variation; and mathematical modeling.

Linear Functions, Equations, and Inequalities

In the grade 7 unit *Moving Straight Ahead*, students learned to recognize, represent symbolically, and analyze relationships in which a dependent variable changes at a constant rate relative to an independent variable. They learned the connections between the rate of change of y, the equation $y = mx + b$, and the slope of the graph. In particular, they learned that m, the coefficient of x in the equation, indicates the constant ratio:

$$\frac{\text{change in } y}{\text{change in } x}$$

which is the slope of the graph of the equation $y = mx + b$.

The constant term b in $y = mx + b$ indicates the y-intercept of the graph. In other words, the graph crosses the y-axis at the point $(0, b)$.

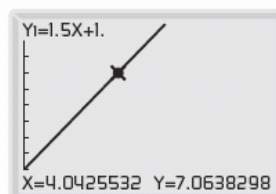
Many questions about linear functions can be answered by solving equations of the form $c = mx + b$ for x (c is a constant). In *Moving Straight Ahead*, students learned to approximate solutions to such equations by using tables and graphs of (x, y) values. They also learned to find exact solutions by undoing the operations to get $x = (c - b) \div m$ and by using the properties of equality (for example, adding the same quantity to both sides of an equation maintains the equality).

Another way to solve $c = mx + b$ for x is to look at the associated fact families. The concept of fact families highlights the inverse relationship between addition and subtraction and between multiplication and division. Students can interpret subtraction problems as missing addend problems and division problems as missing factor problems. Fact families with whole-number operations are introduced in grade 6, and students revisit them in the grade 7 unit *Accentuate the Negative*. The associated addition and subtraction fact family for $c = mx + b$ is $c = mx + b$, $c - b = mx$, and $c - mx = b$. To solve $c = mx + b$ for x, use the equivalent equation $c - b = mx$. Then look at the associated multiplication/division fact family for this equation. We see that $c - b = mx$ is the same as $(c - b) \div m = x$. So $x = (c - b) \div m$.

Although these basic understandings and skills were addressed in *Moving Straight Ahead*, they need to be revisited and practiced to deepen student understanding. The problems in Investigation 2 of *Thinking With Mathematical Models* are designed to promote this sort of review and extension.

Few real problems that involve linear relationships actually call for the kind of precise answers that occur as solutions to equations. Often, problems call for solutions of inequalities of the form $c \le mx + b$ or $c \ge mx + b$. Such inequalities have infinitely many solutions The algebraic, numeric, and graphic strategies that lead to solutions of linear inequalities are related to those for equations. However, there are some key differences. For example, you can multiply or divide both sides of an equation by a negative number without changing the solution. However, when you multiply or divide both sides of an inequality by a negative number, the direction of the inequality is reversed. Consider the inequality $5 < 12$. If you multiply both sides by -1, you must reverse the inequality to get $-5 > -12$.

The treatment of inequalities in this unit is informal. We do not introduce algebraic techniques for solving linear inequalities. That topic is addressed in the upcoming unit *The Shapes of Algebra*. Rather, the problems help invite students to become sensitive to the implications of phrases such as "at least" and "at most," to begin using inequality notation to express problem conditions, and to scan tables and graphs to find solutions for inequalities. For instance, the following table and graph suggest that $1.5x + 1 < 7$ for $x < 4$.



X	Y1	
-1	-.5	
0	1	
1	2.5	
2	4	
3	5.5	
4	7	
5	8.5	

X=4

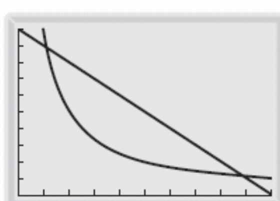
Inverse Variation

The terms *direct variation* and *inverse variation* are used throughout quantitative work in almost every facet of the physical, biological, social, and management sciences. Physicists summarize profound principles with phrases such as, "gravitational attraction of two bodies is directly proportional to the product of their masses and inversely proportional to the square of the distance between their centers of mass."

In everyday language, the phrase "y varies directly with x" means "as x increases, y increases," and "y varies inversely with x" means "as x increases, y decreases." However, the technical meanings of direct variation and inverse variation are more specific than this.

The phrase "y varies directly with x, or is directly proportional to x," means that there is some fixed number k such that $y = kx$. The equation $y = kx$ implies that the ratio $\frac{y}{x}$ is equal to a constant value, k. (This is why the word "proportional" is used in the preceding alternative statement of direct variation.) As long as this proportionality constant k is positive, the pattern of "as x increases, y increases" holds. But there is a particular way that y increases. Specifically, when x increases n times, so does y. So, for example, when x is doubled, y doubles, and when x is tripled, y triples. Note that this is not true for linear relationships of the form $y = kx + b$ for which $b \neq 0$.

An inverse variation is not simply a relationship in which y decreases as x increases. For example, the graphs below show that for both $y = 10 - x$ and $y = \frac{10}{x}$, y decreases as x increases. However, only $y = \frac{10}{x}$ is an inverse variation.



For $y = 10 - x$, y decreases at a constant rate as increasing x-values are subtracted by 10. For $y = \frac{10}{x}$, y decreases at a decreasing rate as 10 is divided by increasing x-values.

For inverse variation, there is a constant k such that $y = \frac{k}{x}$. The inverse variation equation can also be written in the form $xy = k$, which emphasizes that the product of the two variables is constant. In an inverse variation, multiplying x by n multiplies y by $\frac{1}{n}$. So, for example, doubling x halves y, and tripling x multiplies y by $\frac{1}{3}$.

Investigation 3 uses several familiar contexts to develop the concept of inverse variation, building on students' experiences with formulas such as $A = \ell w$ and $d = rt$. The formula $A = \ell w$, for the area of a rectangle, was first explored in the grade 6 unit *Covering and Surrounding*. However, this time, rather than finding the area of a rectangle with a given length and width, as they did in grade 6, students look for combinations of length and width values that give a fixed area. This leads to the formula $\ell = \frac{A}{w}$, which can be used to efficiently calculate such pairs. The formula $d = rt$ relates distance, rate, and time. In earlier units, students calculated the distance traveled for a given rate and time. In this unit, they find combinations of rate and time values that give a fixed distance. This leads to inverse variation equations of the form $r = \frac{d}{t}$ and $t = \frac{d}{r}$, where d is a constant.

Mathematical Modeling

The third key idea of this unit is the notion that the precisely defined objects and operations of mathematics can be used to approximate real-life data patterns that are not as well behaved. In the same sense that a doll or a toy car are models of humans or full-size automobiles, a mathematical function can sometimes be used to model a real-life relationship between variables. For example, the equation $d = 50t$ might make good predictions for the distance traveled as a function of elapsed time. However, this formula will probably not give exact distance values for most specific times, because it is very unlikely that a constant speed can be maintained throughout the trip.

It is important to realize that models are meant to approximate data and may be useful for only a certain range of values. For example, in one ACE exercise, students model the relationship between the age and weight of a Chihuahua based on data for the first few months of a Chihuahua's life. They find that their model is useful for only a limited range of ages because, after a certain age, dogs stop growing.

Using mathematical modeling to solve quantitative problems involves at least five basic steps:

1. Identify the key variables involved in the problem situation.
2. Collect data that indicate the nature of the relationship between those variables.
3. Find an algebraic equation that approximates, or models, the relationship.
4. Use the model to write and solve equations or to make calculations that provide information about values between or beyond the data values.
5. Interpret the results of the mathematical calculations in the context of the original problem.

Effective use of mathematical modeling in solving real problems requires awareness of the overall modeling process and a repertoire of mathematical concepts and skills for model building and analysis. This unit is only a modest introduction to these ideas. It lays a foundation for a more sophisticated and thorough development of modeling strategies in high school and college mathematics and science courses.

Finding a good model for data requires finding a function type with a graph that matches the pattern in a plot of available data. In this unit, students meet only situations in which linear or inverse variation models are particularly appropriate. For the linear examples, students "eyeball" a good fitting line and then use what they know about finding equations for lines to get the function rule. Students do *not* find the "line of best fit" for the data. Finding the line of best fit requires using regression to find the line that *best* matches the data. For the inverse variation examples, we suggest only that students experiment with data plots and test function rules to establish reasonable proportionality constants. Using the plotting and function-graphing capabilities of a graphing calculator makes successive approximation an effective modeling technique. For an introduction to this method, using a TI-83 Plus calculator, see the Technology section on page 10.