Unit Introduction

Stretching and Shrinking

Understanding Similarity

Goals of the Unit

- Identify similar figures by comparing corresponding parts
- Use scale factors and ratios to describe relationships among the side lengths of similar figures
- Construct similar polygons
- Draw shapes on coordinate grids and then use coordinate rules to stretch and shrink those shapes
- Predict the ways that stretching or shrinking a figure affect lengths, angle measures, perimeters, and areas
- Use the properties of similarity to calculate distances and heights that can't be directly measured

Developing Students' Mathematical Habits

The overall goal of the *Connected Mathematics* curriculum is to help students develop sound mathematical practices. Through their work in this unit, students learn important questions to ask themselves about any situation that can be represented and modeled mathematically, such as:

- *What does the everyday use of the word "similar" mean? How does this differ from the mathematical meaning or use of the word?*
- *When two figures are similar, what is the same in each figure? What is different in each figure?*
- *How might we describe these differences?*
- *How do ratios relate to similarity?*
- *When two figures are similar, how can you describe the relationship between their areas? How can you describe the relationship between their perimeters?*
- *In what ways can you apply the ideas about similarity to use in the everyday world?*

Mathematics of the Unit

Pearson Prentice Hall Professional Development

Overview

Knowledge of similarity is important to the development of children's understanding of the geometry in their environment. In their immediate environment and in their studies of natural and social sciences, students frequently encounter phenomena that require familiarity with the ideas of enlargement, scale factors, area growth, indirect measurement, and other similarity-related concepts.

Similarity is an instance of proportionality. For example, if you increase the size of a diagram by 50%, then distances in the enlarged diagram are proportional to distances in the original diagram. Specifically, every distance in the enlargement is a constant multiple (1.5) of the corresponding distance in the original. It is generally understood that understanding proportional reasoning is an important stage in cognitive development.

Students in the middle grades often experience difficulty with ideas of scale. They confuse *adding* situations with *multiplying* situations. Situations requiring comparison by addition or subtraction come first in students' experience with mathematics and often dominate their thinking about any comparison situation, even those in which *scale* is the fundamental issue. For example, when considering the dimensions of a rectangle that began as 3 units by 5 units and was enlarged to a similar rectangle with a short side of 6 units, many students will say the long side is now 8 units rather than 10 units. They add 3 units to the 5 units rather than multiply the 5 units by 2, the scale factor. These students may struggle to build a useful conception that will help them distinguish between situations that call for addition and those that are multiplicative (calling for scaling up or down).

The problems in this unit are designed to help students begin to accumulate the knowledge and experiences necessary to make these kinds of distinctions and to reason about scaling in geometry situations. The next unit, *Comparing and Scaling*, continues to develop these ideas in numerical, rather than geometric, contexts.

Summary of Investigations

Investigation 1

Enlarging and Reducing Shapes

Similarity is introduced at an informal level. Students use their intuition about enlargements and reductions to answer questions. Students make drawings of similar figures using a pair of rubber bands. Then, they compare side lengths, angle measures, perimeters, and areas of the original and enlarged figures.

Investigation 2

Similar Figures

Students build a good working definition of *similar* in mathematical terms. They begin to see connections between geometry and algebra. Using the coordinate system, they draw several geometric figures. Some of the figures are similar to one another and others are not.

They explore algebraic rules that cause images to change size and to move about the coordinate plane. They also compare angle measures and lengths of corresponding sides informally as they investigate transformations. Students find that for two figures to be similar corresponding angles must be congruent and corresponding sides must grow or shrink by the same factor.

Investigation 3

Similar Polygons

Students deepen their understanding of what it means for two figures to be similar. In addition, they explore the relationship between the areas of similar figures. The idea that area does not grow at the same rate as side length when a figure is enlarged is difficult for students to grasp. Through experiments with rep-tiles (shapes where copies are put together to make larger, similar figures), students explore the relationship between the areas of two similar figures. They also discover how triangles are special. These experiences help them build mental images to support their evolving ideas about the relationship between scale factor and area.

Investigation 4

Similarity and Ratios

Students use equivalent ratios to test if figures are similar. They compare ratios of the sides within rectangles (length to width of one rectangle and length to width of the other). Students learn that for non-rectangular shapes such as triangles, you need information about angle measures as well. They learn that between two similar figures, you can find the length of missing sides using either ratios or scale factors.

Investigation 5

Using Similar Triangles and Rectangles

Students apply their knowledge about similarity of triangles to real-world problems. They use the shadow and mirror methods to find the height of a tall object. They compare their data to decide which method gives more consistent results. They also use similar triangles to find the distance across a physical feature, such as a river. In each problem, they find that triangles are similar if their corresponding angles are equal. In the ACE, they use their knowledge about similar rectangles to make similar rectangles and to find missing measurements.

Mathematics Background

The activities in the beginning of the unit elicit students' first notions about similarity as two figures with the same shape. Students may have difficulty with the concept of similarity because of the way the word is used in everyday language—family members are "similar" and houses are "similar." The unit begins by having students informally explore what it means for two geometric figures to be similar. They create similar figures using rubber bands. Early on, they begin to see that some attributes of similar figures are the same while others are not. For example, corresponding angle measures appear to be the same, but corresponding side lengths are different—yet these differences are predictable.

Students' experiences with photocopiers enlarging or shrinking pictures provide another familiar context to begin the exploration of similar figures.

Through the activities in *Stretching and Shrinking*, students will grow to understand that the everyday use of a word and its mathematical use may be different. For us to determine definitively whether two figures are similar, similarity must have a precise mathematical definition.

Similarity

Two figures are similar if:

- the measures of their corresponding angles are equal
- the lengths of their corresponding sides increase by the same factor, called the **scale factor**.

The two Figures A and B below are similar.

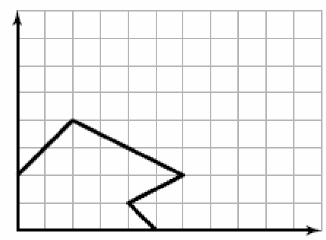


Figure A

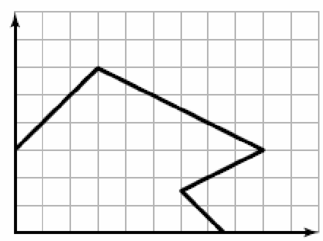


Figure B

The corresponding angle measures are equal. The side lengths from Figure A to Figure B grow by a factor of 1.5. Thus the scale factor from Figure A to Figure B is 1.5. (Figure A *stretches*, or is enlarged.) You can also say the scale factor from Figure B to Figure A is $\frac{1}{1.5}$, or $\frac{2}{3}$. (Figure B *shrinks*, or is reduced.)

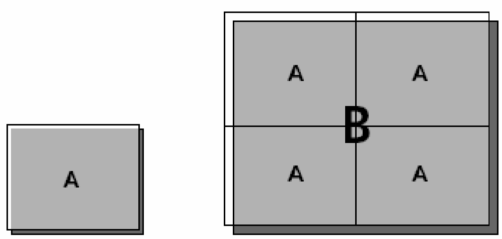
Creating Similar Figures

The rubber-band stretcher introduced in Investigation 1 is a tool for physically producing a similarity transformation. It does not give precise results, but it is an effective way to introduce students to similarity transformations. More precision is gained in transformations using algebraic rules that specify how coordinates change. See Problem 1.2 in the Student Edition for instructions.

In this unit, students make figures on a coordinate system and use algebraic rules to transform them into similar figures. For example, if the coordinates of a figure are multiplied by 2, the algebraic transformation is from (x, y) to $(2x, 2y)$. In general, if the coordinates of a figure are (x, y), algebraic rules of the form $(nx + a, ny + b)$ will transform it into a similar figure with a scale factor of n. These algebraic rules are called *similarity transformations*, which are not introduced as vocabulary in this unit. In the preceding figures, Figure B has been transformed from Figure A by the rule $(1.5x, 1.5y)$.

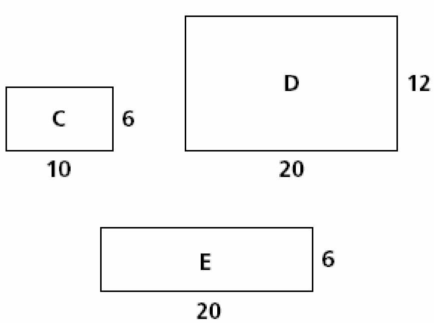
Relationship of Area and Perimeter in Similar Figures

The *perimeters* of one rectangle A and rectangle B below are related by a scale factor of 2. The *area* increases by the square of the scale factor, or 4. This can be seen by dividing rectangle B into four rectangles (labeled A) congruent to rectangle A.



Similarity of Rectangles

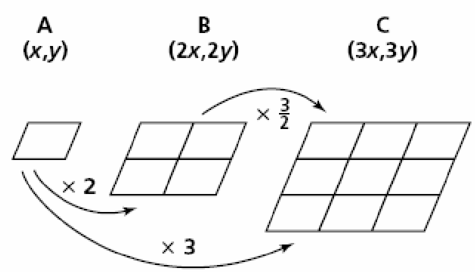
Since all of the angles in rectangles are right angles, you need only check the ratios of the lengths of corresponding sides. For example, rectangles C and D are similar, but neither is similar to rectangle E.



The scale factor from rectangle C to rectangle D is 2 because the length of each side of rectangle C multiplied by 2 gives the length of the corresponding side of rectangle D. The scale factor from rectangle D to rectangle C is $\frac{1}{2}$ because the length of each side of rectangle D multiplied by $\frac{1}{2}$ gives the length of the corresponding side of rectangle C. Rectangle E is not similar to rectangle C, because the lengths of corresponding sides do not increase by the same factor.

Similarity Transformations and Congruence

In general, algebraic rules of the form (nx, ny) are called similarity transformations, because they will transform a figure in the plane into a similar figure in the plane. If the figure described by the rule, (x, y), is compared to the figure described by the rule, (nx, ny), n is the scale factor from the original figure to the image. The scale factor from Figure A (x, y) to Figure B $(2x, 2y)$ is 2. The scale factor from Figure A (x, y) to Figure C $(3x, 3y)$ is 3. This is a special case where $n = 1$ in Figure A. If we compare two figures created by rules when $n \neq 1$ in both figures, then the scale factor is not n. An example is Figure B $(2x, 2y)$ and Figure C $(3x, 3y)$. They are similar to each other. But the scale factor from B to C is $\frac{3}{2}$.



Note that similarity is transitive. If Figure A is similar to Figure B and Figure B is similar to Figure C, then Figure A is similar to Figure C.

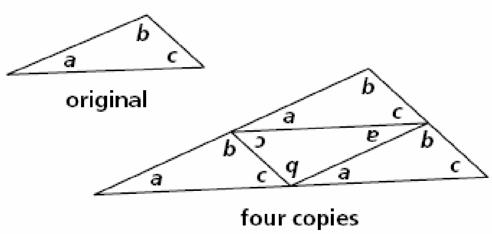
In Problem 2.2, the students will see that adding to x and/or y moves the figure around on the grid, but does not affect its size. This means that a more general form of similarity transformations of this sort is $(nx + a, ny + b)$. Rules of this form, where the coefficient of both x and y is 1 [such as $(x + 3, y - 2)$], move the figure around, but the figure stays exactly the same shape and size (it is congruent to the original).

Congruent is a term from the sixth-grade unit *Shapes and Designs*. Note that the scale factor between two congruent figures is 1. Therefore, congruent figures are also similar.

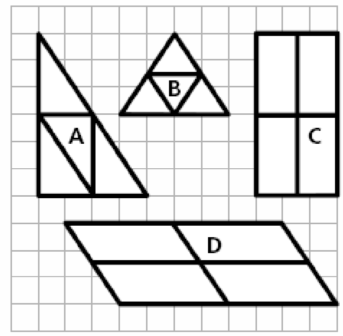
There are other transformations in the plane that preserve congruence, such as flips and turns. These are studied in the eighth-grade unit *Kaleidoscopes, Hubcaps, and Mirrors.*

Comparing Area in Two Similar Figures Using Rep-Tiles

It is generally surprising to students that if you apply a scale factor of 2 to a figure, the area becomes 4 times as large. One approach is to have students calculate the area of a figure and that of its image and compare the results. In the first two investigations, area is explored informally. In Investigation 3, we use rep-tiles to demonstrate that when you apply a scale factor of 2, it requires four copies of the original figure. In this case, you are really measuring area using the original figure as the unit, rather than square inches or square centimeters. If congruent copies of a shape can be put together to make a larger, similar shape, the original shape is called a rep-tile. It takes four congruent triangles to create a larger similar triangle with a scale factor of 2 or nine congruent triangles to create a larger similar triangle with a scale factor of 3. The large triangle below is made from four congruent copies of the smaller triangle. The scale factor from the original triangle to the larger triangle is 2. From the diagram it is fairly easy to see that corresponding angles have equal measures.



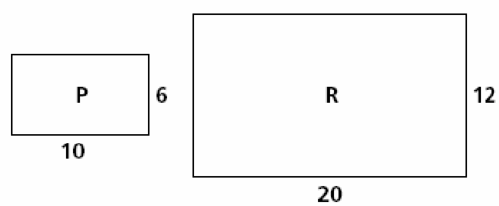
The following examples are also rep-tiles with a scale factor of 2 from the smaller shape to the larger shape.



A misconception that can arise is the idea that tiling is related to similarity. Figures that tile may not make a larger, similar figure. In addition, any figure can be transformed into a larger or smaller image, regardless of whether the figure can tile the plane. Rep-tiles are special because they make area comparisons easy.

Equivalent Ratios

In similar figures, there are several equivalent ratios. Some are formed by comparing lengths *within* a figure. Others are formed by comparing lengths between two figures. For the rectangles below, the ratio of length to width is $\frac{10}{6}$ or $1.\overline{6}$ for rectangle P and $\frac{20}{12}$ or $1.\overline{6}$ for rectangle R.



You can also look at the ratios of corresponding sides *across* two figures. In this situation it is width-to-width and length-to-length. The ratios are $\frac{12}{6}$ and $\frac{20}{10}$, respectively. These ratios are equivalent, and are also equivalent to 2, the scale factor. This second kind of ratio is not formally discussed in the unit, but students have used it informally when they divide corresponding side lengths between two similar figures to find the scale factor. This ratio also appears in an ACE.

The perimeter grows by a factor of 2 and the area grows by a scale factor of 2×2, or 4.

Similarity of Triangles

For polygons other than triangles, you must make sure that the lengths of corresponding sides increase by the same scale factor and that corresponding angle measures are equal when considering similarity.

In *Shapes and Designs*, students explored an important property of triangles—angles determine a triangle's shape. The property leads to the Angle-Angle-Angle Similarity Theorem for Triangles:

> If the measures of corresponding angles in two triangles are equal, then the two triangles are similar.

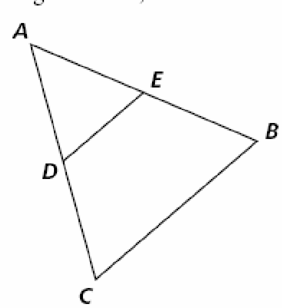
For triangles you only have to check the angles to determine whether two triangles are similar. However, this fact about triangles is only hinted at in the unit. At this stage of their development of understanding of similarity, it is best if students operate with the general definition that applies to all polygons:

> Corresponding angle measures are equal and corresponding side lengths grow by the same scale factor. The next section explains this theorem.

Angle-Angle-Angle Similarity for Triangles

The line that connects the midpoints of two opposite sides of a triangle is parallel to the third side and its length is equal to half the length of the opposite side. The parallel lines create equal corresponding angles.

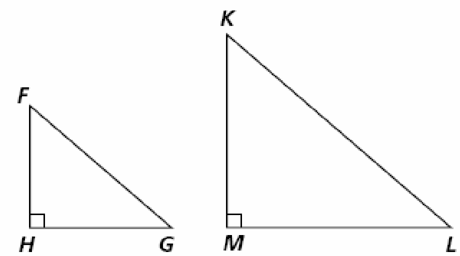
Parallel lines also cut transversals into segments whose ratios are equal. In the following figure, segment DE is parallel to segment CB, so the ratios of the lengths of the segments AD to DC and the lengths of the segments AE to EB are equal.



That is, $\frac{AD}{DC} = \frac{AE}{EB}$.

These facts can be used to show that if the corresponding angles of one triangle are congruent to the corresponding angles of another triangle then the two triangles are similar. This is known as the Angle-Angle-Angle Similarity Theorem. This is true only for triangles. Also, because you know that the angles of a triangle add to 180° you need only check two angles of a triangle in order to verify similarity.

This unit presents an alternative test for similarity. If the corresponding angle measures are equal, then instead of checking the ratio between corresponding sides (the scale factor), you could check the ratios of sides within each figure. Given the two figures below, if $\frac{FG}{GH} = \frac{KL}{LM}$ and $\frac{GH}{FH} = \frac{LM}{KM}$, then the figures are similar.



Solving Problems Using Similar Figures

Equivalent ratios can be used to solve interesting problems. For example, shadows can be thought of as sides of similar triangles because the sunlight hits the objects at the same angle. A building of unknown height and a meter stick, both of which are casting shadows, are shown below. To find the height of the building, you can use the scale factor between the lengths of the shadows. Since going from 0.25 to 10 involves a scale factor of 40, multiply the height of the meter stick by 40 to obtain the height of the building, 40×1 m = 40 m. You could also think of this as $\frac{x}{10} = \frac{1}{0.25}$. Finding the value of x that makes the ratios equivalent gives you the height of the building.