Unit Introduction

Moving Straight Ahead

Linear Relationships

Goals of the Unit

- Recognize problem situations in which two or more variables have a linear relationship to each other
- Construct tables, graphs, and symbolic equations that express linear relationships
- Translate information about linear relations given in a table, a graph, or an equation to one of the other forms
- Understand the connections between linear equations and the patterns in the tables and graphs of those equations: rate of change, slope, and y-intercept
- Solve linear equations
- Solve problems and make decisions about linear relationships using information given in tables, graphs, and symbolic expressions
- Use tables, graphs, and equations of linear relations to answer questions

Developing Students' Mathematical Habits

The overall goal of *Connected Mathematics* is to help students develop sound mathematical habits. Through their work in this and other units about linear relationships, students learn important questions to ask themselves about any situation that can be represented and modeled mathematically, such as

- *What are the variables in the problem?*
- *Do the variables in the problem have a linear relationship to each other?*
- *What patterns in the problem suggest that it is linear?*
- *How can the linear pattern be represented in a problem, in a table, in a graph, or with an equation?*
- *How do changes in one variable affect changes in a related variable?*
- *How are these changes captured in a table, graph, or equation?*
- *How can tables, graphs, and equations of linear relationships be used to express and answer questions?*

Mathematics of the Unit

Pearson Prentice Hall Professional Development

Overview

The primary goal of this unit is for students to develop an understanding of linear relationships or linear functions. Students recognize linear functions by the constant rate of change between two variables in a verbal context, a table, a graph, or an equation. This idea is introduced in Investigation 1 with an experiment in which students determine their walking rates. This experiment is more closely tied to the central idea of constant rate of change between two variables and it provides a "walking rate" theme for the first two Investigations. Identifying, representing, and interpreting linear relationships is the central idea in this unit. Solving linear equations and writing equations for lines is also explored and will be revisited with more complexities in later units—in particular, *Thinking with Mathematical Models* and *Say It With Symbols*.

Summary of Investigations

Investigation 1

Walking Rates

The rates at which students walk and the donation per kilometer that sponsors pay for a walkathon are two contexts for this investigation. Students look at the change in the rate and its effects on various representations. They recognize that graphs of linear functions are straight lines and they begin to see that as the independent variable changes by a constant amount, there is a corresponding constant change in the dependent variable. At this point, some students will *begin* to recognize that the constant change is the coefficient of x in the equation $y = mx + b$.

Investigation 2

Exploring Linear Functions With Graphs and Tables

This Investigation continues the theme of walkathons and helps students deepen their understanding of patterns of change. Constant rate of change and y-intercept are formalized in this investigation.

Students interpret the y-intercept as a special point on a line, an entry in a table, or as the constant b in the equation $y = mx + b$.

They predict constant rate, decide whether relationships are decreasing or increasing, and begin to make connections among points on a line, a pair of data points in a table, and the solution to an equation.

Investigation 3

Solving Equations

Students begin to make the connection among points on a line, pairs of data points in a table, and solutions to equations. They use the properties of equality with equations in pictorial form and transition into solving equations symbolically by adding or subtracting the same number or variable or multiplying or dividing by the same nonzero number or variable on both sides of an equation. They also find the point of intersection of two lines (or the solution of a system of two linear equations) by setting the y values equal and then solving for x.

Investigation 4

Exploring Slope

Students find the ratio of vertical change to horizontal change between two points on a line. The connection between this ratio and constant rate of change is made explicit. Students find the slope of a line given two points on the line. They then find the y-intercept using either a table or graph and write an equation of the form $y = mx + b$, where m is the slope and b is the y-intercept. Then, students apply their knowledge of slope to explore lines that have the same slope (parallel lines) and lines that have slopes that are negative reciprocals (perpendicular lines). Graphing calculators help students explore the slopes of many lines before they make their conjectures.

Mathematics Background

Linear relationships are the "big idea" in this unit. A relationship is linear if there is a constant rate of change between the two variables. That is, for each unit change in x, there is a constant change in y. Throughout the unit, tables, graphs, and equations are used to explore and represent linear relationships as well as solve equations. The mathematics embedded in this unit will be illustrated through several examples similar to the problems in the Student Edition.

Developing the Concept of a Constant Rate or Slope

In the first example, three students determine their walking rates. Alana walks 1 meter per second, Gilberto walks 2 meters per second, and Leanne walks 2.5 meters per second. The distance d that each person walks in t seconds can be represented using an equation.

$d = 1t$ (Alana)
$d = 2t$ (Gilberto)
$d = 2.5t$ (Leanne)

The walking rate is the constant rate of change between the variables distance and time. The three walking rates are shown in the table.

Walking Rates

Time (seconds)	Distance (meters)		
	Alana	Gilberto	Leanne
0	0	0	0
1	1	2	2.5
2	2	4	5
3	3	6	7.5
4	4	8	10
5	5	10	12.5
6	6	12	15
7	7	14	17.5
8	8	16	20
9	9	18	22.5

In the table, the constant rate of change can be observed in the following pattern:

As t increases from 0 to 1 second, d increases by 1 meter for Alana, 2 meters for Gilberto, and 2.5 meters for Leanne.

As t increases from 1 to 2 seconds, d increases again by 1 meter for Alana, 2 meters for Gilberto, and 2.5 meters for Leanne.

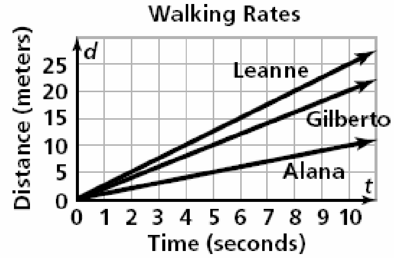
This pattern continues; as t increases by one unit, d increases by a constant amount.

In the symbolic representation, the constant rate of change shows up as the coefficient of t.

$d = 1t$ (Alana)
$d = 2t$ (Gilberto)
$d = 2.5t$ (Leanne)

If we graph the data, the constant rate of change shows up as a straight line.

Walking Rates



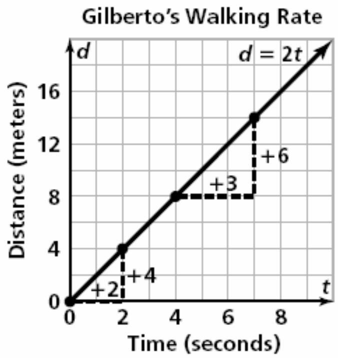
The walking rate of 2.5 meters per second has a line that is steeper than the lines representing the walking rates of 2 meters per second and 1 meter per second.

Connecting Ratio and Rate Concept in Linear Functions

The constant rate of change shown in the table and graph below is called the **slope of the line.**

Gilberto's Walking Rate

	Time (seconds)	Distance (meters)	
	0	0	
+2	1	2	+4
	2	4	
	3	6	
	4	8	
+3	5	10	+6
	6	12	
	7	14	
	8	16	
	9	18	



Slope $= \frac{\text{vertical change}}{\text{horizontal change}}$ for any two points on the line. In the example above, the slope is $\frac{4}{2}$ or $\frac{6}{3}$ or $\frac{2}{1}$.

To help strengthen students' understanding of linear situations, nonlinear situations occur throughout the unit. Most of these occur as tables or graphs like the ones below.

x	*y*
−2	4
−1	1
0	0
1	1
2	4
3	9

x	*y*
−2	3
−1	3
0	3
1	3
2	3
3	3

x	*y*
−5	10
−2	7
−1	4
0	1
1	−2
2	−5

x	*y*
−1	−3
2	3
4	7
8	15
10	19
11	21

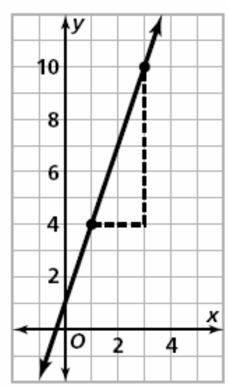
The pattern in the first table is not linear—as x increases 1 unit, there is not a constant rate of change in y. This pattern, which can be represented as $y = x^2$, is studied in Frogs, Fleas, and Painted Cubes. The pattern in the next three tables is linear. The constant rate of change in the first table is 0 and the equation that represents the pattern is $y = 3$. The constant rate of change in the second table is -3 and the y-intercept is 1. Thus, the equation for this pattern is $y = -3x + 1$. The constant rate of change in the last table is 2 and y-intercept is -1, so the equation is $y = 2x - 1$. (Note in the last table, the increments in x are not equal. Students might first try to plot the points to see that they all lie on the same line and then use their understanding of rates and ratios to find the constant rate is 2.) In equations for linear relationships, the exponent of the independent variable x is 1. Since, at this stage, students are not asked to write equations other than linear equations, this distinction is left to *Frogs, Fleas, and Painted Cubes* and *Say It With Symbols*.

Finding the Slope of a Line

In Investigation 4, students are introduced to the idea of slope as a ratio of the vertical change to the horizontal change between two points on a line. The ratio concept of slope is connected to the constant rate of change between two variables. The slope of a line can be found directly from a verbal description, a table, an equation, or by finding the ratio of vertical to horizontal changes between two points on the line.

For example, suppose you are told the points (1, 4) and (3, 10) lie on a line and you are asked to find the slope.

$$\text{slope} = \frac{\text{change in vertical distance}}{\text{change in horizontal distance}} = \frac{10 - 4}{3 - 1} = \frac{6}{2} = \frac{3}{1} = 3$$

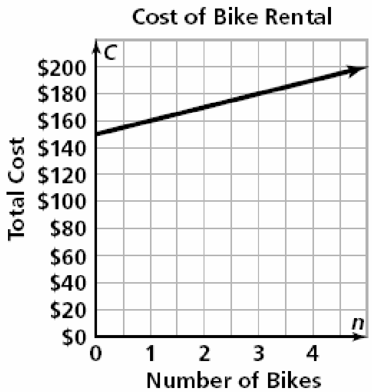


Note the connection to constant rate of change. As x goes from 1 to 3, the (horizontal) change is 2. As y goes from 4 to 10, the (vertical) change is 6.

That is, as x changes by 2 units, y changes by 6 units, or as x changes by 1 unit, y changes by 3 units.

The *y*-Intercept

If you graph $C = 150 + 10n$, you get a straight line.



Notice that the line does not pass through the origin. It crosses the y-axis at $150. This point is called the ***y*-intercept.** The y-intercept is the constant term in the equation $C = 150 + 10n$.

Students often refer to the y-intercept as the "starting point" to generate a table of data for a linear relationship. For example, in a table of the data, (0, 150) is the starting point.

There are many ways to find the y-intercept.

- Use a verbal description.
- Use a graph. It is the point $(0, b)$ on the vertical axis.
- Work backward or forward in a table to find the point $(0, b)$.
- Substitute the slope and the coordinates of one point into the equation $y = mx + b$ and then solve for b. This method is the same as solving a linear equation in one variable.

Solving a Linear Equation

The key to solving equations symbolically is to understand equality. Many students think of equality as a signal "to do something." For example, students encounter the following:

$6 + 15 = \blacksquare$ $\qquad$ $6 \cdot 13 + 15 = \blacksquare$

Students think of an equality as calculating a set of numbers to get an answer. This can be a source of many misconceptions. Instead, equality is a statement that states two quantities are equal. In this unit, students develop understanding of equality. Equality can be thought of as a "balance." To solve an equation means to maintain the equality between two quantities. There are several ways to find the value of a variable in a linear relationship if the value of the other variable is known. For instance, in $C = 150 + 10n$, where C is the rental cost and n is the number of bikes, we might want to know the cost to rent 75 bikes or how many bikes we can rent for $300. In either case, we are solving for one of the variables or we are solving a linear equation with one variable or "one unknown." Students can find the value of the variable by using these methods.

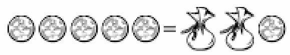
- Solving the equation using symbolic methods.
- Interpreting the information from a table or graph.
- Reasoning about the situation in verbal form—"if it costs $10 per bike, then 75 bikes will cost $750 plus the fixed costs of $150 for a total of $900."

Investigation 3 develops symbolic methods for solving equations. To solve an equation symbolically, we write a series of equivalent equations until we have one in which it is easy to read the value of the variable. Equivalent equations have the same solutions. Equality or equivalence can be maintained by adding, subtracting, multiplying by, or dividing by the same quantity on both sides of the equation. For multiplication and division, the quantity must be nonzero. These procedures are called the **properties of equality**.

To help students develop their understanding of equality, Investigation 3 begins by making the connection to tables and graphs. Linear equations can be solved using tables or graphs. For example, the equation $750 = 150 + 10n$ is associated with the related linear function $y = 150 + 10x$. If $y = 750$, then we are looking for a corresponding value of x. Or, conversely, we can find a corresponding value of y if we are given a value of x. At first we expect that students will use graphs or tables and, in some cases, simply substitute values into the equation and calculate the missing value.

Once the concept and properties of equality have been explored, though, we move to a pictorial situation to develop a symbolic method for solving linear equations. First, students explore adding, subtracting, multiplying, or dividing the same number to both sides of a numeric sentence: $85 = 70 + 15$.

Next, students explore equations like the following with coins and pouches.



Each pouch in an equation must have the same number of coins and the number of coins on both sides of the equality sign are equal.

Students can solve this by taking 1 coin from each side. This leaves 4 coins on the left and 2 pouches on the right. Because each pouch must have the same number of coins, students can intuitively divide both sides by 2 to find that each pouch contains 2 coins. This provides a transition to the more abstract situations of solving linear equations in one unknown. Students first find the number of coins using the pictures. Then they translate each picture into a symbolic statement. For example, if x represents the number of coins in a pouch, then the preceding pictorial statement can be represented as $5 = 2x + 1$. Next, students apply the properties of equality to isolate the variable—that is, they solve the equation for x. The set of equations is selected carefully to allow students an opportunity to look at what each symbolic statement means.

In this unit, we solve the following types of equations:

$$6 - 3x = 10$$
$$5 + 17x = 12x - 9$$
$$2(x + 3) = 10$$

Integers and the distributive property were developed in *Accentuate the Negative*. Review of integers and the distributive property are provided in the Connections section of each ACE.

Solving a System of Two Linear Equations

Students informally solve systems of linear equations throughout the unit. They use graphs and tables to find the point of intersection of two lines. For example, a problem in Investigation 1 has students comparing pledge plans. Students are asked to determine if any of the two pledge plans will have the same amount of money.

$$C_{\text{Leanne}} = 10$$
$$C_{\text{Gilberto}} = 2x$$
$$C_{\text{Alana}} = 5 + 0.5x$$

C represents the amount of money collected for x kilometers.

To find the number of kilometers that make two plans equal, students start by using tables or graphs. Later, students can solve two of the preceding equations symbolically. To find when Gilberto's plan equals Alana's plan, they write $2x = 5 + 0.5x$ and solve for x.

This can be done for either of the two plans. From the graphs of the three plans, students note that all three plans will never have the same costs for a given amount of kilometers. In Investigation 3, students learn that situations like this one represent a system of linear equations. Without calling attention to it, students solve a linear system by finding the point of intersection of the two lines symbolically. They find the x value when the y values are equal or by setting the two equations equal. Students can also use a table or graph to find the point of intersection.

It is important that students understand what a solution is, whether they are dealing with a symbolic solution or a graphical solution, and that they can connect these two views of a solution. Thus, $x = 3.\overline{3}$ is a solution for $2x = 5 + 0.5x$. It means that if Gilberto and Alana each walk $3.\overline{3}$ kilometers, they earn the same amount. Graphically, the lines $y = 2x$ and $y = 5 + 0.5x$ intersect at $(3.\overline{3}, 6.\overline{6})$. This solution means that when they each walk $3.\overline{3}$ kilometers, they each earn about \$6.67.

Ideas about inequality are informally explored by asking questions like: "if $x = 4$, does Gilberto earn more or less than Alana?" Students can answer this question by finding the missing coordinate in $(4, \blacksquare)$ for each equation, $y = 2x$ and $y = 5 + 0.5x$ and then comparing the coordinates. In the *Shapes of Algebra* unit, students explore linear inequalities and systems of linear equations in depth.

Finding the Equation of a Line

In this unit, most of the linear equations are of the form $y = mx + b$. In some situations, equations can be obtained by translating the verbal situation directly into symbols.

To find the equation of a line symbolically, we find the slope of the line (coefficient of x) and the y-intercept (the constant term b) and substitute these values into the equation $y = mx + b$.

Students are given one of the following pairs of information about a line:

- The slope and y-intercept
- The slope and a point on the line
- Two points on a line

In the first case, the values are substituted directly into the equation $y = mx + b$. In the second case, the slope is substituted into the equation $y = mx + b$ and the coordinates of the point are substituted into the equation to find the y-intercept. In the third case, the slope is determined using two points and then another point is used to find the y-intercept. The following methods can be used to find the y-intercept of a line.

Method 1: Finding the *y*-Intercept Symbolically

First, find the slope of the line.

Given the two points, (1, 4) and (3, 10), the slope is 3.

Then substitute the slope into the equation $y = mx + b$ to obtain $y = 3x + b$. Since both points lie on the line, the x and y values of the points satisfy the equation. Choose one point [(1, 4)] and substitute it into the equation.

$$4 = 3(1) + b$$

Solving for b, $b = 1$.
Now substitute for b.

$$y = 3x + 1$$

This is the equation of the line that passes through the points (1, 4) and (3, 10).

Method 2: Finding the y-Intercept Using a Table

Some students prefer to find the y-intercept using a table and working backwards. Consider the following table.

x	y
3	7
4	10
5	13
6	16
7	19

First, students note that this is a linear relationship because as x increases by 1 unit, y increases by 3 units. So, the slope is 3. To find the y-intercept, they work backwards using the slope. That is, as x decreases by 1 unit, y decreases by 3 units. This is repeated until $x = 0$.

x	y
0	−2
1	1
2	4
3	7
4	10
5	13

The y-intercept is -2 and the equation of the line is $y = 3x - 2$. Some students have a strong sense of the y-intercept as a starting point, to which you repeatedly add a constant number to generate the table.

Method 3: Finding the *y*-Intercept Using a Graph

Students also use a graph to find the y-intercept. They extend the line to intercept the y-axis. Some use the ratio definition of slope to work from a point on the graph backward or forward until they hit the y-axis.