

Unit Introduction

Filling and Wrapping Three-Dimensional Measurement

Goals of the Unit

In *Filling and Wrapping*, we will explore surface area and volume of objects, especially rectangular prisms, cylinders, cones, and spheres. The unit should help students to

- Understand volume as a measure of *filling* an object and surface area as a measure of *wrapping* an object
- Design and use nets to visualize and calculate surface areas of prisms and cylinders
- Explore patterns among the volumes of cylinders, cones, and spheres
- Develop strategies for finding the volumes of square pyramids, prisms, cylinders, cones, and spheres directly and by comparison with known volumes
- Understand that three-dimensional figures may have the same volume but quite different surface areas
- Understand how changes in one or more dimensions of a rectangular prism or cylinder affects the prism's volume and surface area
- Extend students' understanding of similarity and scale factors to three-dimensional figures
- Use surface area and volume to solve a variety of real-world problems

Developing Students' Mathematical Habits

Throughout their work in this and other geometry units, students learn to ask important questions about volume and surface area.

- *What quantities are involved in the problem?*
- *Which measures of an object are involved—volume or surface area?*
- *Is an exact answer required?*
- *What method should I use to determine these measures?*
- *What strategies or formulas might help?*

Overview

In *Filling and Wrapping*, students explore the surface areas and volumes of rectangular prisms and cylinders in depth. They look informally at how changing the scale of a box affects its surface area and volume. They also informally investigate other solids—including cones, spheres, and square pyramids—to develop volume relationships.

Summary of Investigations

Investigation 1

Building Boxes

Students are introduced to the ideas of volume and surface area through the concepts of wrapping and filling, building on their knowledge of area and perimeter of two-dimensional figures from the *Covering and Surrounding* unit. Rectangular prisms are described by their dimensions: length, width, and height.

Investigation 2

Designing Rectangular Boxes

Students continue their exploration of surface area and investigate its relationship to volume. The terms *surface area* and *volume* are introduced as vocabulary. By thinking about filling boxes in layers, students develop the formula for volume of a rectangular prism. The volume of a box is the number of blocks in the bottom layer multiplied by the number of layers—the area of the base times the height of the prism. (This strategy holds for all prisms.)

Investigation 3

Prisms and Cylinders

Students compare the volumes and surface areas of a variety of prisms with regular bases and a common height. Students build prisms by folding several sheets of congruent rectangular paper into the shapes of triangular, rectangular, and

hexagonal prisms. They find the volume of any rectangular prism by determining how many unit cubes would fill the prism. They observe that the volume of a prism increases as the number of lateral sides increases. The volume of a cylinder is the area of the base (the number of unit cubes in the bottom layer) of the cylinder multiplied by its height (the number of layers). Surface area is informally looked at as the sum of the area of the bases (circles) and lateral side (rectangle).

Prisms and cylinders come together as students design a rectangular box with the *same* volume as a given cylinder. They find the surface area of the box is greater than the surface area of the cylinder.

Investigation 4

Cones, Spheres, and Pyramids

Students compare the volumes of cones, cylinders, and spheres in an application. Students determine how many times the volume of the cone or sphere will fill the cylinder and then look for relationships among the three volumes. (Finding the surface areas of cones and spheres is not considered in this unit.) Students also compare the volume of a square pyramid to a cube.

Investigation 5

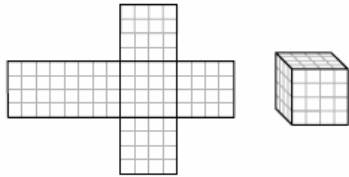
Scaling Boxes

Students study the effects of changing the dimensions or the volume of a rectangular prism in the context of designing compost containers. They explore two central ideas: how to double the volume of a rectangular prism and examine how other measures change as a result, and the effects of applying scale factors to the dimensions of rectangular prisms. What effect does doubling (tripling, quadrupling, etc.) each dimension of a rectangular container have on its volume and surface area? Students apply their knowledge of similarity and scale factors to explore the relationships between a model of a cruise ship and the actual cruise ship. This last problem connects many of the ideas discussed in this unit.

Mathematics Background

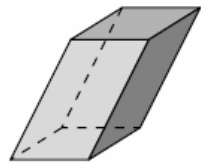
Rectangular Prisms

Students begin the unit by exploring the surface area of a rectangular box. The strategy for finding the surface area of a box is to determine the total area needed to wrap the container. Students create nets that can be folded into boxes. The area of the net becomes the surface area of the box.



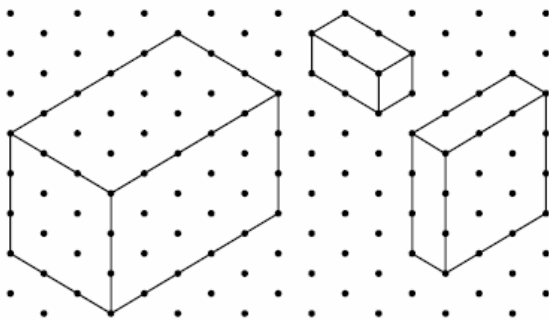
This provides a visual representation of surface area as a two-dimensional measure of a three-dimensional object.

In the student edition, a rectangular prism is defined as a three-dimensional shape with six rectangular faces. Technically, this defines a **right rectangular prism**. An **oblique rectangular prism** also has opposite sides that are rectangles, but at least two opposite sides must be nonrectangular parallelograms.



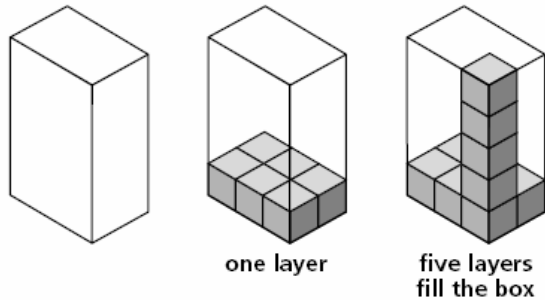
Oblique rectangular prism

In this unit we only discuss right rectangular prisms. There is one ACE question on oblique rectangular prisms. The following figures are right rectangular prisms drawn on isometric dot paper.



Isometric drawings are another useful 2-dimensional representation of 3-dimensional objects. You may wish to have isometric dot paper available throughout the unit for students to use in making sketches of the 3-dimensional prism they create.

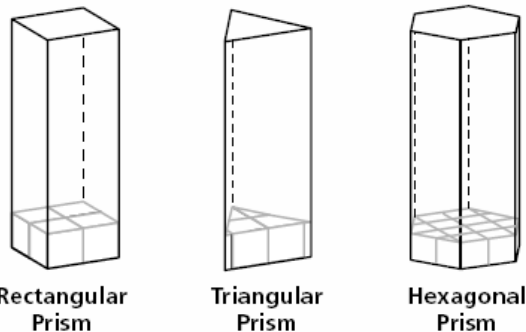
The strategy for finding the volume of a rectangular box is to count the number of layers of unit cubes it takes to fill the container. The number of unit cubes in a layer is equal to the area of the base—one unit cube sits on each square unit in the base. The volume (the total number of unit cubes) of a rectangular prism is the area of its base (the number of unit cubes in the first layer) multiplied by its height (the total number of layers).



one layer

five layers fill the box

The same layering strategy is used to generalize the method for finding the volume of any prism. The volume of any prism is the area of its base multiplied by its height.



Rectangular Prism

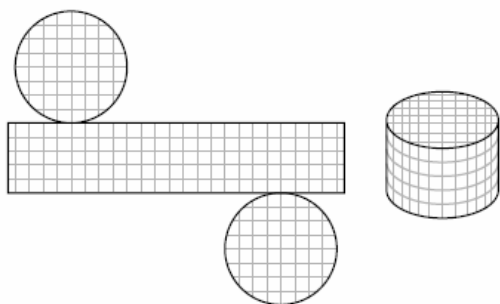
Triangular Prism

Hexagonal Prism

Students also informally compare the volume of two rectangular boxes by filling one box with rice or sand and then pouring the sand into the other rectangular box.

Cylinders

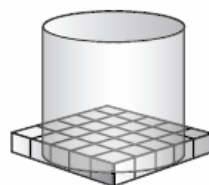
The surface area and volume of a cylinder are developed in a similar way. Like a prism, a cylinder has two identical faces (circles). Also like a prism, a cylinder has a lateral surface that flattens to a rectangle. Students will notice all of these similarities. If they need language for the rectangle that is part of the net of both cylinders and prisms, feel free to introduce the term *lateral surface*. The term is not included in the student materials. Students cut and fold a net to form a cylinder. In the process, they find that the surface area of the cylinder is the area of the rectangle that forms the lateral surface plus the areas of the two circular ends.



Cylinders can be thought of as circular prisms. In this case, it is easy to extend the techniques for measuring prisms to techniques for measuring cylinders. The volume of a cylinder is developed as the number of unit cubes in one layer (the area of the circular base) multiplied by the number of layers (the height) needed to fill the cylinder. Because the edge of the circular base intersects the unit cubes, students will have to estimate the number of cubes in the bottom layer.

From the *Covering and Surrounding* unit, students know the formula for the area of a circle. They can apply this formula to find the area of the base of a cylinder. The area of the base is multiplied by the height to find the volume.

The volume of a cylinder = $\pi r^2 h$. Students investigate rectangular prisms with polygonal bases. If the prisms have the same heights, then as the number of sides of the polygonal base increases, the shape gets closer to a cylinder.



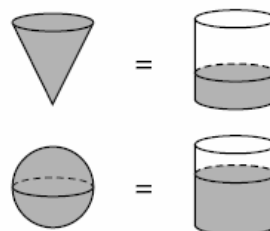
Estimate the number of unit cubes in one layer.



And multiply by the number of layers.

Cones, Spheres, and Rectangular Pyramids

Students conduct an experiment to demonstrate the relationships among the volumes of a cylinder, a cone, and a sphere. If all three have the same radius and the same height (the height being equal to two radii), then it takes three cones full of sand to fill the cylinder, and one and a half spheres full of sand to fill the cylinder.



These relationships may also be expressed as follows:

volume of the cone = $\frac{1}{3}$ of the volume of the cylinder or $\frac{1}{3}\pi r^2 h$

volume of the sphere = $\frac{2}{3}$ of the volume of the cylinder or $\frac{2}{3}\pi r^2(2r)$ or $\frac{4}{3}\pi r^3$

For cones and spheres, only the volume is studied. Surface area in these two cases is not considered here because the reasoning needed would take us too far afield. Formulas for these are sometimes considered within the context of high school geometry or calculus courses.

The volume of a square pyramid is found in a similar way by comparing it to a square prism. This is easily generalized to finding the volume of a rectangular pyramid. If the base of the pyramid is a polygon, then as the number of sides of the polygon increases, the shape of the pyramid gets closer to a cone.

Relationship Between Surface Area and Fixed Volume

Students also investigate the effects of a change in dimension, surface area, or volume on the other attributes of a three-dimensional object. For example, if 24 unit cubes are arranged in a rectangular shape and packaged in a rectangular box, which arrangement of the cubes will require the least (the most) packaging material? By physically arranging the blocks and determining the surface area of each arrangement, students discover that a column of 24 cubes requires the most packaging, and the arrangement that is the most like a cube (2 by 3 by 4) requires the least amount of packaging. This is similar to ideas students have studied about plane figures: For a fixed area, the rectangle that is most like a square has the least perimeter of any rectangle with the same area. A similar relationship holds for a fixed surface area. The rectangular prism that is the most like a cube will have the greatest volume for a fixed surface area.

In fact, it is not cube-ness that minimizes surface area, but sphere-ness. For a fixed volume, a sphere has the smallest surface area and, conversely for a fixed surface area, a sphere has the largest volume.

Effects of Changing Attributes—Similar Prisms

Through the context of designing an indoor compost box, students explore the effects that changing a box's dimension have on the volume and surface area of the box. Given the dimensions of a compost box known to decompose a half pound of garbage per day, students investigate what size box would decompose one pound of garbage per day. They find that they need to

double only one dimension of a rectangular box to double its volume.

Students also look at the effects of doubling all three dimensions of a box. Making scale models of the original box and the new box helps students visualize the effect of the scale factor. Doubling each dimension of a rectangular prism increases the surface area by $2 \times 2 = 4$ times (a scale factor of 2^2) and volume by $2 \times 2 \times 2 = 8$ times (a scale factor of 2^3). The surfaces of the two prisms are similar figures with a scale factor of 2 from the small prism to the large prism. This exploration connects back to ideas in the similarity unit, *Stretching and Shrinking*.

When we describe a cylinder, we generally give only two dimensions: the height and the radius. The radius is constant in every direction, so we need not give a "length radius" and a "width radius." Yet, when we change the radius, we change both the length and the width of the base. That is, we change two dimensions, not just one. In the ACE for Investigation 5, students investigate similar cylinders. At that point, students may need to discuss whether a cylinder is 2- or 3-dimensional in this sense.

Measurement

All measurements are approximations. In the work in this unit, this idea will become more apparent than usual. Students' calculations of surface area and volume will often involve an approximation of the number π , and they will often use a calculated amount as a value in a subsequent calculation. Be aware that although students' answers will often differ, the answers may reflect correct reasoning and correct mathematics.