Unit Introduction

The Shapes of Algebra

Linear Systems and Inequalities

Goals of the Unit

- Write and use equations of circles
- Determine if lines are parallel or perpendicular by looking at patterns in their graphs, coordinates, and equations
- Find coordinates of points that divide line segments in various ratios
- Find solutions to inequalities represented by graphs or equations
- Write inequalities that fit given situations
- Solve systems of linear equations by graphing, by substituting, and by combining equations
- Choose strategically the most efficient solution method for a given system of linear equations
- Graph linear inequalities and systems of inequalities
- Describe the points that lie in regions determined by linear inequalities and systems of inequalities
- Use systems of linear equations and inequalities to solve problems

Developing Students' Mathematical Habits

The overall goal of the *Connected Mathematics* (CMP) curriculum is to help students develop sound mathematical habits. Through their work in this unit, students should ask themselves questions about situations that involve algebra, such as:

- *What patterns relate the coordinates of points on lines and curves?*
- *Does the problem involve an equation or an inequality?*
- *Does the problem call for writing and/or solving a system of equations? If so, what method would be useful for solving the system?*
- *What patterns relate the points whose coordinates satisfy linear equations?*
- *Are there systematic methods that can be used to solve any systems of linear equations?*

Mathematics of the Unit

Pearson Prentice Hall Professional Development

Overview

Algebraic ideas and techniques are powerful tools for reasoning about geometric shapes on a coordinate grid. Conversely, geometric images are useful aids to algebraic reasoning about linear equations and inequalities. This unit is designed to capitalize on the strong connections between algebra and geometry in order to extend students' understanding and skill in several significant aspects of those two key strands in the middle grades curriculum.

Circles, triangles, rectangles, and general parallelograms are visually familiar shapes. When those shapes are drawn on a coordinate grid, the coordinates of points comprising the figures can be identified as solutions for equations. The special geometric relationships that characterize various figures can be described and analyzed by studying properties of the corresponding equations.

For instance, points on a circle with center at the origin and radius r will have coordinates satisfying the equation:

$$x^2 + y^2 = r^2.$$

Points on (non-vertical) sides of any polygon will have coordinates satisfying linear equations of the form:

$$y = mx + b.$$

The coefficient m indicates the slope of the line, so parallel sides have the same slope and perpendicular sides have slopes with product -1.

Reasoning in the opposite direction—from geometry to algebra—the relationship between linear functions and straight-line graphs is helpful in understanding solution possibilities for systems of linear equations. If two lines such as $3x + 2y = 5$ and $3x + 2y = 7$ are parallel but disjoint, the corresponding system of linear equations will have no solutions. If the lines intersect in a single point, as do $x + 2y = 4$ and $-x + 3y = 6$, which meet only at $(0, 2)$, the corresponding system will have a unique solution.

When problem conditions suggest constraints represented by linear inequalities in two variables such as $3x + 2y < 5$ or $3x + 2y > 7$, the graphic representation becomes a half plane with many solutions.

These are the key ideas developed by problems in *The Shapes of Algebra* unit. They extend earlier work with the Pythagorean Theorem by connecting it to the standard equation for circles; with properties of polygons by connecting parallel and perpendicular lines to slopes of lines and linear functions; and with solutions of linear equations by considering solutions of linear systems and equations in standard $ax + by = c$ form, and solutions of linear inequalities.

These topics are standard parts of traditional Algebra I syllabi and they are included in many standard algebra examinations. However, the *Connected Mathematics* approach to the topics exploits the rich connections between algebra and geometry to strengthen student understanding of problems and solution methods that are often taught and learned in quite formal and rote ways. Since students who proceed to mathematically oriented academic specialties will undoubtedly study these topics in greater detail in high school mathematics, it is important to develop the sort of conceptual understanding that will provide a solid base for future work, not simply to settle for short-term rote learning of procedures that will be quickly forgotten.

Summary of Investigations

Investigation 1

Equations for Circles and Polygons

The main goal of this investigation is to begin to explore the geometry of the coordinate plane. The set of three problems uses the geometry of crop circle designs to pique student interest in use of coordinates and algebraic equations to describe geometric shapes. When one looks at the fantastic crop circle designs that have been created and impressed on fields all over the world, it is impossible to imagine that this work was done freehand and without use of coordinate locators and equations.

The equation for a circle, midpoints of lines, and parallel and perpendicular lines are explored in this investigation.

Investigation 2

Linear Equations and Inequalities

The main goal of this investigation is to lay out informal and graphic foundations for dealing with systems of linear equations in two variables. It reverses the geometry–algebra connection story line by beginning with algebraic expressions and using geometry to picture the conditions implied. It also focuses on equations in the $y = mx + b$ form and inequalities, $ax + b < cx + d$.

Investigation 3

Equations With Two or More Variables

The main goal of this investigation is to focus attention on what is curiously referred to as *standard form* ($ax + by = c$) of linear equations, their graphs, and the geometric interpretation of solving a system of such equations. The investigation lays a visual foundation for the more algorithmic methods of solving linear systems that are addressed in Investigation 4. Students use graphical methods to solve systems of linear equations in standard form, or they can solve for y in each equation and set the two resulting equations equal.

Investigation 4

Solving Systems of Linear Equations Symbolically

The aim of this investigation is to develop understanding and skill in use of several standard strategies for finding solutions by algebraic reasoning, such as solving a system of linear equations by substitution or by combining equations.

Investigation 5

Linear Inequalities

This investigation develops some of the basic techniques for work with linear inequalities in the form $ax + by < c$ and $ax + by > c$. It does so in the context of problems that continue the unit perspective on connecting algebraic ideas to geometric shapes by emphasizing the connection between linear inequalities and the half-planes that are their graphs.

Mathematics Background

There are several key mathematical concepts developed in this unit.

Equations of Circles

To find the distance between two points where the line segment connecting the points is neither horizontal nor vertical, it is possible to treat the line segment as the hypotenuse of a right triangle. Problems like these were done in the unit *Looking for Pythagoras*. You can find the length of the hypotenuse—and thus the distance between the points—using the Pythagorean Theorem.

The following is a formula for finding the distance between two points,(x_1, y_1) and (x_2, y_2) in the plane:

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

The distance formula is simply a different form of the Pythagorean Theorem, $a^2 + b^2 = c^2$. In this case, leg a is the horizontal distance between the points $x_1 - x_2$. Leg b is the vertical distance between the points $y_1 - y_2$. The hypotenuse is the distance we seek. Thus, $a^2 + b^2 = c^2$ becomes: $(x_1 - x_2)^2 + (y_1 - y_2)^2 = d^2$.

Taking the square root of both sides yields the distance formula

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = d$$

Since the distance formula is an application of the Pythagorean theorem, either formula can be used to determine the equation for a circle. The use of each formula is explained below.

A circle can be defined as the set of points a given distance from a fixed point, called the center. When a circle of radius r is drawn on a coordinate grid with center at the origin (see Figure A), any point on the circle (x, y) can be viewed as a vertex of a right triangle with legs of length x and y and hypotenuse of length r (see Figure B). The Pythagorean Theorem says that for right triangles, $a^2 + b^2 = c^2$, where a and b are the legs of the right triangle and c is the hypotenuse. Applying the Pythagorean Theorem to the right triangle drawn in Figure A, we have $x^2 + y^2 = r^2$. This equation would be true for any point (x, y) on the circle in Figure A. So the equation represents the relationship for all points on the circle.