Unit Introduction

How Likely Is It?

Understanding Probability

Goals of the Unit

- Understand that probabilities are useful for predicting what will happen over the long run
- Understand the concepts of equally likely and not equally likely
- Understand that a game of chance is fair only if each player has the same chance of winning, not just a possible chance of winning
- Understand that there are two ways to build probability models: by gathering data from experiments (experimental probability) and by analyzing the possible equally likely outcomes (theoretical probability)
- Understand that experimental probabilities are better estimates of theoretical probabilities when they are based on larger numbers of trials
- Develop strategies for finding both experimental and theoretical probabilities
- Interpret statements of probability to make decisions or answer questions

Developing Students' Mathematical Habits

The overall goal of the *Connected Mathematics* curriculum is to help students develop sound mathematical habits. Through their work in this and other probability units, students learn important questions to ask themselves about any situation that can be represented and modeled mathematically, such as

- *What are the possible outcomes that can occur for the events in this situation?*
- *How could I determine the experimental probability of each of the outcomes?*
- *Is it possible to determine the theoretical probability of each of the outcomes?*
- *If so, what are these probabilities?*
- *How can I use the probabilities I have found to answer questions or make decisions about this situation?*

Mathematics of the Unit

Pearson Prentice Hall Professional Development

Overview

This is the first unit in the *Connected Mathematics* curriculum that will develop students' abilities to understand and reason about probability. Students will gain an understanding of experimental and theoretical probabilities and the relationship between them. The unit also makes important connections between probability and rational numbers, geometry, statistics, science, and business.

Questions about how likely an event is are asked every day. Such questions ask about the probability of an event happening, and the answers are important to many people. This unit explores different types of probability questions in contexts that are interesting to students, such as games, advertising, contests, and genetics.

Summary of Investigations

Investigation 1

A First Look at Chance

Investigation 1 introduces students to experimental probabilities and the idea of the chances that some event will occur. Students will have many opportunities to collect data through experimentation using such items as coins and paper cups. Then, they will use the data to assign experimental probabilities to the results. This investigation also introduces students to the notion of equally likely outcomes and that the range of probabilities for a situation is 0 to 1.

Investigation 2

Experimental and Theoretical Probability

This investigation continues students' work with finding experimental probabilities and formally introduces the term *theoretical probability*. The colors of identically-shaped objects chosen from a bag are analyzed both theoretically and experimentally. Students also consider the difference between a particular outcome being *possible* and being *likely* (or probable) and determine if a game or probabilistic situation is fair. Making an organized list or tree diagram are two strategies for finding all the theoretical outcomes.

Investigation 3

Making Decisions With Probability

Investigation 3 introduces spinners as a new context for thinking about probabilities. The crucial difference between spinners and the other objects studied so far is that a spinner has a continuous range of possibilities by subdividing the 360° angle at the end of a spinner into any number of angles. Students also analyze various methods for making fair decisions and devise a simulation to find experimental probabilities.

Investigation 4

Probability, Genetics, and Games

Investigation 4 gives students opportunities to apply and further develop their knowledge about probability in a variety of interesting situations, including applications of probability in genetics and games. Genetics and games can be linked by using a table to find theoretical probabilities of outcomes.

Mathematics Background

The Meaning of Probability

The terms *chance* and *probability* apply to situations that have uncertain outcomes on individual trials but a regular pattern of outcomes over many trials. For example, when you toss a coin, you are uncertain whether it will come up heads or tails. But you do know that over the long run, if it is a fair coin, you will get about half heads and half tails. This does not mean you won't get several heads in a row, or that, if you get heads now, you are more likely to get tails on the next toss. Uncertainty of an individual outcome but predictable regularity in the long run is a difficult concept for students to grasp. It often takes a significant amount of time and a variety of experiences that challenge prior conceptions before students understand this basic concept of probability.

If you toss a tack into the air, you know that the tack will land either on its head or its side. If you toss the tack many times, you can use the ratio of the number of times the tack lands on its side to the total number of tosses to estimate the likelihood that the tack will land on its side. Since you find this ratio through experimentation, the ratio is called an *experimental probability*.

The experimental probability that the tack will land on its side can be expressed as:

$$P(\text{side}) = \frac{\text{number of times the tack lands on its side}}{\text{total number of tosses}}$$

Many uses of probability in daily life are based on experimental probabilities. You collect data for a large number of trials and observe the frequency of a particular result. This is the *relative-frequency interpretation* of probability. The probability that it will rain or that Shaquille O'Neal will make a free throw are two uses of experimental probabilities based on relative frequencies.

Another way to determine probability is to find the *theoretical probability* of a situation. For example, you can examine the theoretical probability of a fair coin landing heads or tails by analyzing the situation. If you toss a fair coin, you know that it will land either heads up or tails up and that each outcome is *equally likely*. Since there are two possible equally likely outcomes, the probability of a fair coin landing heads up is 1 of 2, or $\frac{1}{2}$.

You can write this statement as $P(\text{heads}) = \frac{1}{2}$. In general, the theoretical probability that a coin will land heads up can be expressed as:

$$P(\text{heads}) = \frac{\text{number of favorable outcomes}}{\text{number of possible outcomes}}$$

$$= \frac{1 \text{ (heads)}}{2 \text{ (number of outcomes)}}$$

Another example of a theoretical probability that occurs in this unit involves the roll of a number cube. When you roll a number cube, there are six possible outcomes: 1, 2, 3, 4, 5, and 6.

Each outcome is equally likely on any roll of a number cube. Thus, $P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$. If you roll a number cube 36 times, you can expect each number to occur about 6 times. You can use this theoretical probability to make an estimate: If a number cube is rolled many times, you can expect each number to occur about $\frac{1}{6}$ of the time. You can also compute the probability of events that include more than one equally likely outcome. Take the following question, for example:

- What is the theoretical probability of rolling a multiple of 3 on a number cube?

Since two of the six equally likely outcomes, 3 and 6, are multiples of 3, the probability of a multiple of 3 occurring is $\frac{2}{6}$, or $\frac{1}{3}$.

Some important aspects of the concept of probability are illustrated below using the action of rolling a number cube.

- A probability is a number that is less than or equal to 1 and greater than or equal to 0.
- The sum of the probabilities of all outcomes is equal to 1. In the case of rolling a number cube there are six outcomes, each with a probability of $\frac{1}{6}$. The sum is:

  $P(1) + P(2) + P(3) + P(4) + P(5) + P(6) = 1.$
- Sometimes problems involve a probability that an outcome A *or* B will occur. For example, to find the probability that the number occurring is either 1 *or* 6, you consider 1 and 6 as favorable outcomes, so

  $P(1 \text{ or } 6) = P(1) + P(6) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6}$, or $\frac{1}{3}$.
- Sometimes problems may involve a probability that outcomes A *and* B occur. For example, the probability that a number is greater than 3 *and* is even, means that the number must be both. The probability can be thought of as the intersection of the set of numbers that are greater than 3 with the set of even numbers up to 6. Thus,

  $P(> 3 \text{ and even}) = P(4) + P(6) = \frac{2}{6}$, or $\frac{1}{3}$.
- The probability of rolling the number 7 is $\frac{0}{6}$, or 0. It is an impossible outcome.
- The probability of rolling a 1, 2, 3, 4, 5, or 6 is $\frac{6}{6}$, or 1. This outcome is certain.
- The probability of rolling a number that is *not* 6 is $\frac{5}{6}$, or $P(\text{not } 6) = P(1, 2, 3, 4, 5, \text{or } 6) - P(6) = 1 - \frac{1}{6} = \frac{5}{6}$.

Strategies for Finding Outcomes

In many situations, making an organized list can help you determine all the possible outcomes. In the situation of rolling a number cube, there are only six outcomes to list. Some situations involve more than one action. For example, suppose you toss a fair coin twice. How many possible outcomes are there? You can list the outcomes as they come to mind, but it is often more efficient to generate the outcomes in a systematic way. This helps to ensure that you find all the possible outcomes. In the situation of tossing a coin twice, you can list the possibilities for the first toss, namely heads (H) or tails (T). Suppose the first toss resulted in heads (H). What can happen next?

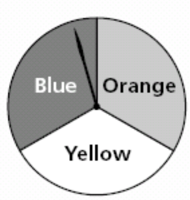
First Toss	Second Toss
H	H
H	T
T	H
T	T

Since the two tosses of the coin are *independent* (the results of one do not affect or depend on the other), you have two possible outcomes (H or T) for the second toss. Thus, you can have either HH or HT. Now suppose the first toss resulted in tails (T). Again there are two possible outcomes for the second toss, H or T. In this case, you can have either TH or TT. Thus, there are four possible outcomes when you toss a coin twice. Since you have considered all the possibilities in a systematic way, you can feel confident that you have found all the possible outcomes.

NOTE: When students toss two coins at once, they may perceive only one way to get a no-match: one head and one tail. Because the two coins are the same, students may not see heads-tails as being different from tails-heads. One way to address this is to toss one coin twice, paying attention to the *order* that the heads and tails come up. Not all students will see this as being the same as tossing two coins at the same time. Another way to investigate the question is to have the coins have different years stamped on them. A 1975 head and 1982 tail are different from a 1975 tail and 1982 head.

Tree Diagrams and Organized Lists

Tree diagrams, introduced in Investigation 2, offer students another way to systematically determine all the possible outcomes in a situation. For example, suppose you spin the pointer of a spinner with three sections (made by three angles with the same measure) and you roll a number cube.

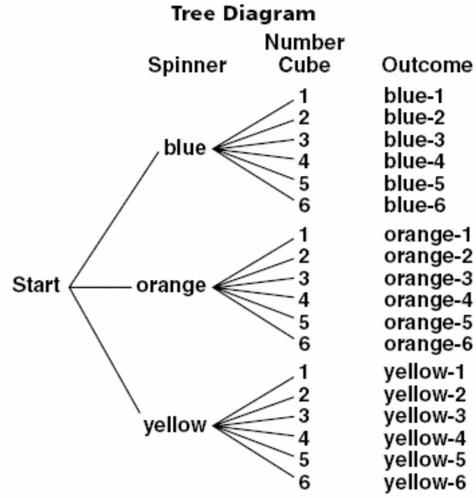


An organized list or a tree diagram can be used to determine all the possible outcomes.

Organized List

Color	Number Cube	Outcome
blue	1	blue-1
blue	2	blue-2
blue	3	blue-3
blue	4	blue-4
blue	5	blue-5
blue	6	blue-6
orange	1	orange-1
orange	2	orange-2
orange	3	orange-3
orange	4	orange-4
orange	5	orange-5
orange	6	orange-6
yellow	1	yellow-1
yellow	2	yellow-2
yellow	3	yellow-3
yellow	4	yellow-4
yellow	5	yellow-5
yellow	6	yellow-6

Tree Diagram



There are 18 equally likely outcomes, so the probability of each outcome is $\frac{1}{18}$.

In this unit, students use tree diagrams to find the number of equally likely outcomes in situations with a large number of possible outcomes. Tree diagrams are particularly useful for listing outcomes in situations involving a series of actions, such as rolling a number cube twice, tossing a coin four times, or choosing several items from a menu, such as a sandwich, drink, and dessert. Tree diagrams can be used as a basis for understanding the multiplication of probabilities, though they are not intended to be used that way in this unit. Students do not yet understand enough about probability to know when and why it is appropriate to multiply probabilities. For example, in the preceding example, the probability of spinning a blue is $\frac{1}{3}$ and the probability of rolling a 1 is $\frac{1}{6}$. The probability of spinning a blue *and* rolling a 1 is $\frac{1}{3} \times \frac{1}{6}$, or $\frac{1}{18}$.

Experimental vs. Theoretical Probability

In some situations, it is easier to find theoretical probabilities. In others, it is easier to find experimental probabilities. For example, in this unit students will find experimental probabilities for a tossed paper cup landing on its side or on one of its ends. They will not be able to determine the theoretical probabilities. Although "end" and "side" are the possible outcomes, they are not necessarily equally likely.

Probabilities are useful for predicting what will happen over the long run, yet a theoretical or experimental probability does not tell us exactly what will happen. For example, if you toss a coin 40 times, you may not get exactly 20 heads. However, if you toss a coin 1,000 times, the fraction of heads will be fairly close to $\frac{1}{2}$.

Comment on the Likelihood of 20 Heads

The string HHHHHHHHHHHHHHHHHHHH is just as likely as any other string of 20 tosses, such as HTTHHHHTTHTHTHTHTHTT. Since there are two choices for the first position, two for the second position, and so on for all 20 positions, there are $2 \times 2 \times 2 \times 2 \ldots$ (a product of 20 factors of 2), or 2^{20}, different strings that can occur.

A string of 20 heads is one of these 1,048,576 possible strings. Notice that there is only one way to have 20 heads in a row, but there are over a million ways to have a mixture of heads and tails. For example, there are 184,756 ways to get 10 heads and 10 tails. Hence, having some mixture of heads and tails is much more likely than having 20 heads, because there are more ways to arrange them. Still, any one *specific* arrangement of 10 heads and 10 tails is just as likely (or unlikely) as a string of 20 heads. The probability of each specific string is $(0.5)^{20}$, or $\frac{1}{1{,}048{,}576}$ (roughly one in a million).

Comment on Random

In mathematics, *random* has a meaning somewhat different from its everyday usage. In everyday English, random is often used to mean *haphazard* and completely unpredictable, in either the long or short term. In mathematics, random means that any particular outcome is unpredictable, but the long-term behavior is quite stable. When you toss a coin, it is random because you never know whether the next toss will be heads or tails, but you do know that in the long run you will have close to 50% heads.

Comment on the Use of Outcomes, Results, and Events

Mathematically, an *outcome* is one of the possible results of an experiment or event. Mathematicians also use the term *event* to mean outcome. For example, consider the probability of heads occurring when a coin is tossed. In this situation, heads occurring is the event or outcome. In this unit, we have chosen the language of *outcome* because it seems to be more intuitive for both teachers and students. On occasion, *result* is used if it is more natural. For most purposes, outcome and result are interchangeable terms.

Law of Large Numbers

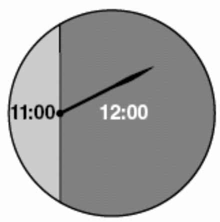
Experimental data gathered over many trials should produce probabilities that are close to the theoretical probabilities. This idea is sometimes called the *Law of Large Numbers*. If you can calculate a theoretical probability, you can use it to predict what will happen in the long run rather than having to rely on experimentation. The Law of Large Numbers applies to mathematically random outcomes.

It is important to understand what the Law of Large Numbers says, as well as what it does not say. It does *not* say that you should expect exactly 50% heads in any given large number of trials. Instead, it says that as the number of trials gets larger, you can expect the percent of heads to be around 50%. For 1 million tosses, exactly 50% (500,000) heads is improbable. But for 1 million tosses, it would be *extremely* unlikely for the percent of heads to be less than 49% or more than 51%.

Comment on Area and Angles in a Spinner

The probabilities in a spinner are determined by the relative measures of the *angles* in each section rather than by their *areas*. The two are interchangeable in the spinners of Investigation 3 because the center of the spinner is in the center of a circle. The areas of the sections vary in proportion to their angles.

In the spinner below, however, the two outcomes are still equally likely although their areas are not the same. The angles taken up by each section equal 180°.



Comment on Experimental and Theoretical Probabilities in Genetics

In genetics, the difference between experimental and theoretical probabilities is less clear-cut. Because the basic experiments in genetics deal with reproduction, these experiments can involve causative factors that cannot always be anticipated. You can, of course, study the results of many experiments by studying populations. You can think of the experiment as choosing someone at random. Then, if you know the characteristics of the population as a whole, you have a theoretical probability (just as you did when you knew how many red and blue blocks were in a bag).

We have deliberately chosen not to use the language of *theoretical* and *experimental* with genetics in Investigation 4 in order to avoid confusion. If students raise the question, you might discuss what the experiment is in each case, and the basis for the theoretical probability.

Using Probabilities to Make Predictions and Decisions

Once you have a probability (theoretical or experimental), you can use it to make predictions. For example, if a coin is tossed 1,000 times, you can predict that heads will occur about 500 times. If you roll a number cube 1,000 times, you can predict that a 3 occurs about $\frac{1}{6}$ of the time, or about $\frac{1}{6} \times 1{,}000 \approx 167$ times. Another way to think of this is as equivalent fractions: $\frac{1}{6} \approx \frac{167}{1{,}000}$. Note that $\frac{167}{1{,}000}$ is approximately equal to $\frac{1}{6}$.

Students often seek ways to make decisions that are fair. For example, how can you select students for a field trip that can accommodate only ten students? To be fair, the method chosen should give each student an equal (or the same) chance of being chosen. Also, students sometimes find themselves in situations where they would like to know the probabilities of a favorable outcome, such as rubbing two spots on a card containing five spots. Some of the spots, if rubbed, lead to prizes. Such situations can be simulated by an experiment such as choosing colored marbles from a bag. Knowing the probability in these situations can help make decisions about whether to play the game or predict genetic traits.