

Unit Introduction

Bit and Pieces III

Computing With Decimals and Percents

Goals of the Unit

- Build on knowledge about operations with fractions and whole numbers
- Develop and use benchmarks and other strategies to estimate the answers to computations with decimals
- Develop meaning of and algorithms for operations with decimals
- Use the relationship between decimals and fractions to develop and understand why decimal algorithms work
- Use the place value interpretation of decimals to make sense of shortcut algorithms for operations
- Generalize number patterns to help make sense of decimal operations
- Choose between addition, subtraction, multiplication, or division as an appropriate operation to use to solve a problem
- Understand that decimals are often associated with measurements in real-world situations
- Solve problems using operations with decimals.
- Use understanding of operations and the meaning of percents to solve percent problems of the form $a\%$ of b equals c for any one of the variables a , b , or c
- Create and interpret circle graphs

Developing Students' Mathematical Habits

The overall goal of the *Connected Mathematics* (CMP) curriculum is to help students develop sound mathematical habits. Through their work in this and other number units, students should ask themselves questions about situations that involve decimals and percents:

- *What is the whole (unit) in this situation?*
- *How big are the numbers in this problem?*
- *About how large will the sum (difference, product, or quotient) be?*
- *How do these decimals compare to fractions that I know?*
- *Why are percents useful in this problem?*

Overview

Rational numbers and their various forms of representation and interpretation are the heart of the middle-grades experiences with number concepts. In earlier units, students have explored various meanings of and models for rational numbers in fraction form and in decimal form. They have developed efficient algorithms for addition, subtraction, multiplication, and division with fractions. These algorithms will be one basis on which understanding of operations with decimals will be built. Here the meaning of decimals as special fractions with denominators that are powers of 10 will be the focus and help make the connection.

In elementary school, the place value chart is often extended to include digits to the right of the decimal point. However, this knowledge is very fragile for most students in grade six. Using a place value interpretation alone places a great deal of faith in number patterns to help students make sense of operations, especially of multiplication and division. We have chosen instead to use both a fraction interpretation and a place value interpretation of decimals to support the development of algorithms.

This unit is designed to provide experiences in building algorithms for the four basic operations with decimals, as well as opportunities for students to consider when such operations are useful in solving problems. For example, what signals indicate to the student that division will help solve a problem? Building this kind of thinking and reasoning supports the development of skill with the algorithms. We build on students' familiarity with money as an entry point and use other familiar measurement situations where measures are given in decimals.

As the title also implies, this unit uses the students' knowledge of operations with decimals to return to percents and to further develop students' understanding and skill in solving percent problems. Particular attention is paid to solving the relationship *a% of b equals c*, when only two of the three values *a*, *b*, and *c* are given. In many texts, the three cases for the missing variable are taught as separate, unrelated problems. We want students to see that finding a percent of a number, finding what percent one number is of another number,

and finding the original number if you know a percent of the number are all versions of this basic relationship with different unknown variables. In the case of finding a percent of a number, the unknown is *c*, for example, 5% of 24 equals *c*. In finding what percent one number is of another, the unknown is *a*, for example, *a%* of 48 equals 6. And, in finding the original number if you know the percent of the number, the unknown is *b*, for example, 15% of *b* equals 27.

Before we look at the specific mathematical ideas developed and used in this unit, it is helpful to review the ideas developed in *Bits and Pieces I* and *II* as these are the underpinnings for this unit.

CMP and Fractions:

Review of *Bits and Pieces I* and *II*

In CMP, we have developed a set of connected units that comprise the rational number strand for grade six. In *Bits and Pieces I*, the first unit on fractions, decimals, and percents, the investigations ask students to make sense of the meaning of fractions, decimals, and percents in different contexts. The unit emphasizes developing an understanding of basic interpretations, models, equivalence, and ordering of rational numbers. Students learn to move among equivalent forms of fractions and to move among fractions, decimals, and percents. They also build benchmarks for estimating locations of rational numbers on a number line and begin to estimate simple sums and differences. The models introduced and used in *Bits and Pieces I* (fraction strips, bars, number lines, and area models) are continued and built upon in *Bits and Pieces II*.

In *Bits and Pieces II*, students develop algorithms for fraction computations. As is the case with other aspects of CMP, students are confronted with situations that call for putting together, taking apart, duplicating, counting an array, sharing, grouping, partitioning, measuring, etc. As they confront such situations, they not only learn to "do" addition, subtraction, multiplication, and division of fractions, but they also learn the meaning of the operations and the kinds of situations that call for each. As students work individually, in groups, and as a whole class on problems, they learn ways of thinking about and operating with fractions and they practice the algorithms to develop skill in carrying them out.

We expect students to finish *Bits and Pieces II* knowing algorithms for computation that they understand and can use with facility.

Interpretations of Fractions

The major interpretations of fractions students encounter in *Bits and Pieces I* and *II* are:

1. fractions as parts of a whole
2. fractions as measures or quantities
3. fractions as indicated division

Interpretations such as fractions as operators (“stretchers” or “shrinkers”) and fractions as rates, ratios, or parts of a proportion are foreshadowed here and continued in later grades. For a fuller discussion of these ideas, please see the Teacher Guides to *Bits and Pieces I* and *II*.

Models of Fractions

The models of rational numbers used throughout CMP were chosen because they connect directly to important interpretations of rational numbers. The fraction models used for developing both meaning and the operations on them are:

1. fraction-strip models
2. number-line models
3. grid-area models
4. partition models

For a fuller discussion of these models please see the Teacher Guides to *Bits and Pieces I* and *II*.

Summary of Investigations

Investigation 1

Decimals—More or Less!

Investigation 1 develops addition and subtraction of decimals. One problem focuses on estimation strategies, as do other problems in the unit. The initial questions CMP helps students to ask are, “About how great will the answer be? What makes sense?” These give students a way to know if their computations, done by hand or by calculator, are at least close to the correct answer or obviously wrong. In many situations an estimate is sufficient to “solve” the problem or make the needed decision.

Other problems in Investigation 1 focus on the place value interpretation of a number and what that means for adding or subtracting numbers.

Addition-subtraction fact families are used to help solve for missing addends or sums in situations written in symbolic form. Students write mathematical sentences using symbols to indicate the required computation(s). An underlying goal of all this work is learning both to write and to read mathematical language. Additionally, students learn the value of changing the representation of fractions and decimals that they need to add or subtract to a form with common denominators, so that the numerators can be added or subtracted. Students’ previous work in locating and representing fractions on the number line is critical to the development of common denominators as a strategy for adding or subtracting. In the end, students articulate an algorithm for adding and subtracting.

Investigation 2

Decimal Times

This investigation focuses on developing an algorithm for multiplying decimals. Students use fractions to help make sense of multiplication of decimals. They look at products, find missing factors, and use estimation as a way to determine where the decimal has to be in a product of decimal numbers. Problem 2.4 lays the groundwork for the simple shortcut algorithm: Multiply the decimals as whole numbers and adjust the place of the decimal in the product.

Investigation 3

The Decimal Divide

Investigation 3 develops an algorithm for division of decimals. In developing the algorithm, students solve a set of contextualized problems that provide a common sense way to think about decimal division based on what they already know about whole-number and fraction division. Students use the fraction form of decimals to develop an algorithm for dividing decimals. The last two problems look at patterns in division and in terminating and repeating decimals.

Investigation 4

Using Percents

In all three problems of Investigation 4, students look at real situations in which one encounters percents. The typical situations of discounts, taxes, and tips help students think about taking a percent of a number. The discount and tax situations help students to consider the amount left when a reduction is made and the total when taxes are added.

Investigation 5

More About Percents

In this investigation, students are asked to devise a general strategy for finding a percent when they are dealing with totals that are more than or less than 100.

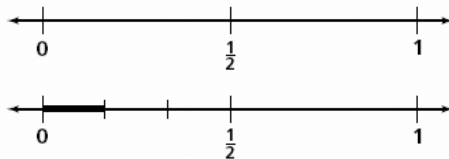
Mathematics Background

The following are key ideas in developing algorithms for fractions that are continued in *Bits and Pieces III*.

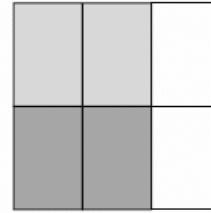
Decimal Multiplication and Division

One of the first hurdles for students in their understanding of multiplication of fractions and decimals is realizing that multiplication does not always “make greater.” Their experience with whole-number multiplication has firmly established this incorrect belief. In fact, multiplication involving numbers that are not whole numbers may be interpreted as an operator that may “stretch” (make greater) or “shrink” (make less) depending on whether the fraction or decimal is greater or less than one. This is a big idea that supports understanding what multiplication of fractions or decimals entails.

Models for multiplication of fractions used in the *Bits and Pieces II* unit are both partitioning and area models. An example of partitioning is: Seth is running $\frac{1}{3}$ of a $\frac{1}{2}$ -mile relay race. How far will he run?



An example of an area model is: Mr. Sims asks to buy $\frac{1}{2}$ of a pan that is $\frac{2}{3}$ full. What fraction of a whole pan does Mr. Sims buy?



These models continue to be useful with decimals.

Division also has its share of conceptual difficulties. The answer to a division problem involving fractions, whether in fraction or decimal form, is not necessarily less than the dividend. Again, it depends on the size of the divisor. For example, the answer to $3 \div 0.25 = 12$ is greater than the dividend. Another example is $0.25 \div 0.4 = 0.625$. What matters is that the divisor is less than 1.

Decimal Estimation

In *Bits and Pieces I* we looked at benchmark fractions and their decimal equivalents. These fraction benchmark ideas can be used to estimate small decimal computations as well. For example, let us look at estimating the sum, the difference, the product, and the quotient of 0.78 and 0.14.

$0.78 + 0.14$ is near $\frac{3}{4} + \frac{1}{8}$ or $\frac{6}{8} + \frac{1}{8}$. So a reasonable estimate is a little less than 1 or perhaps about 0.9.

$0.78 - 0.14$ would be about $\frac{6}{8} - \frac{1}{8}$ or $\frac{5}{8}$. So a reasonable estimate is about 0.6.

0.78×0.14 would be about $\frac{6}{8} \times \frac{1}{8}$ or $\frac{6}{64}$. So a reasonable estimate is about 0.1.

$0.78 \div 0.14$ would be about $\frac{6}{8} \div \frac{1}{8}$ or about 6.

In each case estimating with benchmark fractions is helpful.

Other strategies are also useful. In the examples above, students can round to convenient decimals in their heads and estimate from these. For example, you could round 0.78 to 0.8 and 0.14 to 0.1. This rounds one number up and the other

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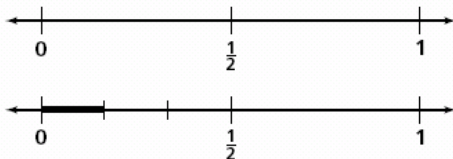
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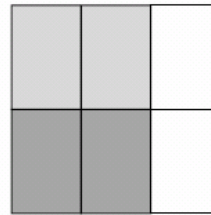
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$0.78 \div 0.14$ would be about $\frac{6}{8} \div \frac{1}{8}$ or about 6.

In each case estimating with benchmark fractions is helpful.

Other strategies are also useful. In the examples above, students can round to convenient decimals in their heads and estimate from these. For example, you could round 0.78 to 0.8 and 0.14 to 0.1. This rounds one number up and the other

down. This would give the following mental computations with one-digit decimals to make an estimate:

$0.78 + 0.14$ is near $0.8 + 0.1$ but a bit greater than 0.9 .

$0.78 - 0.14$ would be near $0.8 - 0.1$ but a bit less than 0.7 .

0.78×0.14 would be about 0.8×0.1 or a bit greater than 0.08 . So you might guess that the product is closer to 0.1 .

$0.78 \div 0.14$ would be about $0.8 \div 0.1$ or about 8 .

Since the dividend was rounded up and the divisor was rounded down, the estimate will be too large. So we might estimate the answer to be closer to 6 .

There is no “one right way to estimate” that works for every situation. Students need to build a repertoire of strategies and a sense, through experience, discussion, and analysis, of what works in a given situation.

Developing Algorithms for Computing With Decimals

As we stated earlier, students have two ways of making sense of what decimals mean. Students extend the place value system on which our number system is built, or they can interpret decimals as fractions. (Obviously these two ideas are related, but they have different looks and feels to students.) In order to have the most robust understanding of and skill with computation, students need to understand each of these meanings of decimals and be able to use them. Depending on the operation, the fraction interpretation or the place value interpretation may contribute more directly to finding shortcut algorithms. However, looking at the algorithms developed through each lens can help develop deeper understanding.

The location of a digit in a number shows the value of the digit. This is a fundamentally important concept for students. Without place value understanding, work with decimals will suffer. In the Getting Ready for Problem 1.2, students look at why it makes sense to add by examining the value of each digit and recognizing that you must be careful to add digits that represent comparable values.

Looking at the patterns in such problems as the following, (Problem 2.4) brings a *place value*

perspective to developing an algorithm for multiplication of decimals.

1. Find the following products using the fact that $21 \times 11 = 231$.

a. 2.1×11 b. 2.1×1.1

c. 2.1×0.11 d. 2.1×0.011

e. 0.21×11 f. 0.021×1.1

g. 0.021×0.11 h. 0.21×0.011

2. Test the algorithm you wrote in Question C on these problems.

We continue to use a combination of place value and fraction interpretations of decimals to develop a division algorithm. We also help students relate division of decimals to the long form of division of whole numbers.

Decimal Forms of Rational Numbers

Students have already observed that the decimal forms for some fractions, such as $\frac{1}{3} = 0.3333333\dots$, “go on forever,” but show a repeating pattern. Non-repeating infinite decimals such as $0.101101110111011110\dots$ *never* reach a point where the digits start to repeat. These are *irrational numbers*, such as π and $\sqrt{2}$, and they are not considered in this unit. They are discussed in the grade 8 unit, *Looking for Pythagoras*. Here we are interested in the decimal forms of *rational numbers*.

Some rational numbers have a finite (or terminating) decimal form. Here are some examples: $\frac{1}{2} = 0.5$, $\frac{3}{4} = 0.75$, $\frac{1}{8} = 0.125$, $\frac{3}{25} = 0.12$. Others have an infinite repeating decimal form, such as $\frac{2}{3} = 0.66666666\dots$, $\frac{8}{15} = 0.5333333$, or $\frac{3}{7} = 0.42857142857142\dots$

In Problem 3.5, we examine rational numbers to figure out how to predict whether a given fraction will have a repeating or terminating decimal form. Rational numbers in *simplified fraction form* that have only 2’s or 5’s in the prime factorization of the denominator will have a terminating decimal form, for example, $\frac{12}{75} = 0.16$. In simplified fraction form, $\frac{12}{75} = \frac{4}{25}$, which has only factors of five in the denominator. Fractions with factors other than 2’s and 5’s in the simplest denominator equivalent form will have a repeating decimal form, for example $\frac{13}{75} = 0.17333333\dots$ and $\frac{4}{3} = 1.333333\dots$

Percents

This unit uses the students' knowledge of operations with decimals to return to the uses of percents and further develops students' understanding and skill in solving percent problems. Using the relationship $a\%$ of b equals c , any one of the letters a , b , or c can be the missing value. This means that the three kinds of percent problems developed separately in some texts are looked at in context with the focus on the relationships among these three variables. The following are examples of problems students solve in the fourth investigation of this unit.

1. Jill wants to buy a CD that is priced at \$7.50. The sales tax is 6%. What will be the total cost of the CD?

In this problem, students know the price of the item and need to find 6% of the price. So here the values of a and b are known and the students must find the value of c . The equation would look like the following:

$$6\% \text{ of } \$7.50 = c.$$

They have to multiply 0.06×7.50 to find c .

2. Customers leave Jerome \$2.50 as a tip for service. The tip is 20% of the total bill for their food. How much is the bill?

In the equation we now have 20% of b equals \$2.50 and we have to find the value of b . Solving this equation can be done in several ways. One way is to ask how many 20%'s it takes to make 100%. In this case we need five. So $5 \times \$2.50$ gives us \$12.50. Later in CMP, students will have more sophisticated equation-solving techniques and will be able to think of solving a problem like this by dividing each side by 0.2 to get $b = \$12.50$.

3. At another music store, Rita gets a \$12 discount off a purchase of \$48. What percent discount does she get?

In this situation our equation looks like this: $a\%$ of \$48 equals \$12. We have to find the percent that 12 is of 48. Students can informally solve this by asking themselves how many 12's it takes to get 48. It takes four 12's to make 48 so the percent must be $\frac{1}{4}$ of 100%. This would be 25%. As with the previous example, students will have more sophisticated solution methods later in the CMP curriculum

and can return to these types of equations and divide each side of the equation by 48 to find 0.25, or 25%, as the answer.

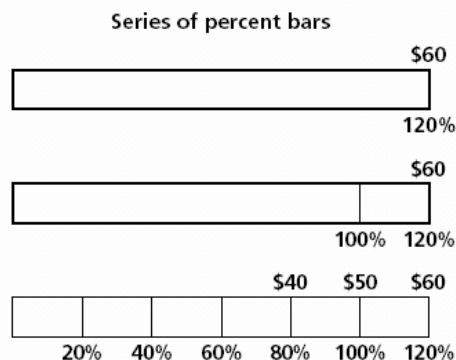
These informal equation-solving techniques are powerful ways of thinking that are based on understanding the situation. Work of this kind should lead to better student monitoring of their work when equation-solving techniques are more fully developed in grades 7 and 8. Rushing to techniques here may mask understanding of the problem situations and what the problem is asking.

Working Backwards

A final kind of percent situation to which students are introduced is set in a restaurant scenario in which you know the total amount of money you have. You know what percent the taxes and the tip are, but you want to figure out how much you can spend on food and still pay the bill. Here is a specific example: Your group has \$60. The tax on food is 5% and you want to leave a 15% tip on the food before tax. How much can you spend on food?

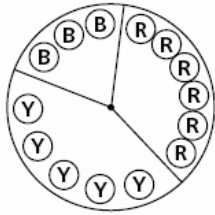
The first thing to realize is that you can add together the 5% and the 15% since you are taking these percents of the same number and adding them together ($0.15x + 0.05x = 0.20x$). If the tip were also on the tax, the situation would be more complicated.

Here is a diagram that illustrates a way of solving the problem. Since the tax and tip will be 20% of the cost of the meal, the total amount can be thought of as representing 120% and the meal cost representing 100%. This leads to partitioning the 120% bar into six equal parts, each of which is 20%. On the cost line, the six partitions show us that each 20% represents \$10, so the cost of the food must not exceed \$50.



Circle Graphs

The final problem in the unit uses what students know about percents to make a new form of graphical representation for data called a circle graph. The key to a circle graph is that you know there are 360° in a full turn around the center of a circle. To represent the data, you need to figure out what angle represents the amount of turn for a certain percent of the data. One way to help students understand circle graphs is to start with small disks or other small objects, such as peas that have been colored. The total for the colors should be in different proportions. Form a circle with the objects making sure to put all the objects of one color adjacent to each other. Then draw the outline of the circle around the objects and draw lines from the center of the circle to separate the segments made by each color. This makes a rough circle graph.



There are 14 disks in the circle. Count the number of each color. Use fractions to represent the fractional part of the data each color represents.

$$\frac{3}{14} \text{ blue; } \frac{5}{14} \text{ yellow; } \frac{6}{14} \text{ red}$$

Use the equivalent percents to multiply by 360° .

$$\frac{3}{14} \approx 21\% \text{ and } 0.21 \times 360^\circ = 75.6^\circ$$

$$\frac{5}{14} \approx 36\% \text{ and } 0.36 \times 360^\circ = 129.6^\circ$$

$$\frac{6}{14} \approx 43\% \text{ and } 0.43 \times 360^\circ = 154.8^\circ$$

Now make the circle graph precisely.

