Unit Introduction

Bits and Pieces II

Understanding Fraction Operations

Goals of the Unit

- Use benchmarks and other strategies to estimate the reasonableness of results of operations with fractions
- Develop ways to model sums, differences, products, and quotients with areas, strips, and number lines
- Use estimates and exact solutions to make decisions
- Look for and generalize patterns in numbers
- Use knowledge of fractions and equivalence of fractions to develop algorithms for adding, subtracting, multiplying, and dividing fractions
- Recognize when addition, subtraction, multiplication, or division is the appropriate operation to solve a problem
- Write fact families to show the inverse relationship between addition and subtraction, and between multiplication and division
- Solve problems using arithmetic operations on fractions

Developing Students' Mathematical Habits

The overall goal of *Connected Mathematics* is to help students develop sound mathematical habits. Through their work in this and other number units, students learn important questions to ask themselves about any situation that is represented and modeled mathematically, such as:

- *What kinds of models can be used to show computation with fractions?*
- *Will the strategies and algorithms we have developed apply to all fractional quantities?*
- *What do whole number operations reveal about the meaning of operations with fractions?*
- *Do results from algorithms support those found with the models?*
- *How can estimation help in this situation?*

Mathematics of the Unit

Pearson Prentice Hall
Professional
Development

Overview

The overall goal of *Bits and Pieces II* is to develop meaning for and skill with computations involving fractions. When students finish this unit they should know algorithms for computations that they understand and can use with ease. This unit does not teach a specific or preferred algorithm for working with rational numbers. Instead it helps the teacher create a classroom environment in which students work on problems and generate ideas and strategies that make sense to them. At a point in the development of each operation, students are asked to pull together their strategies into an algorithm that works for all fraction situations involving that operation. As they work individually, in groups, and as a whole class on the problems, they develop and practice the algorithms to develop skill in carrying them out. This development process allows students to recognize special cases that can be easily handled and yet ends with students having an efficient general algorithm that works for all cases within an operation.

Letting the students wrestle with making sense of situations may take more time in the beginning. However, the payoff in the long run is that students learn to think and to reason about mathematical situations and although the algorithms need practice, they will not need to be taught repeatedly. The invented algorithms of students are often efficient and can evolve into standard algorithms. As they do, students understand why standard algorithms work.

We expect that when students finish this unit they will have an understanding of the meaning for computations with fractions. Students should be able to decide which operation is appropriate in a given situation. In addition, students should be able to use number sense, benchmarks, and operation sense to estimate solutions for computational situations as well as use estimation to decide if exact answers are reasonable.

Summary of Investigations

Investigation 1

Estimating With Fractions

Investigation 1 focuses on estimating sums of fractions and decimals. It builds on work with benchmarks, ordering, and the relationship between decimals and fractions in *Bits and Pieces I*. Students play a game in which they estimate the size of sums. Students also explore underestimation and overestimation as a strategy to reason about contextual situations where an estimate is needed and it is important to know if the estimate is above or below an exact amount.

Investigation 2

Adding and Subtracting Fractions

This investigation focuses on developing computational understanding and skill in adding and subtracting fractions. This investigation does not *give* students algorithms for computation. Instead, it prepares students to figure out how to add and subtract fractions by emphasizing flexibility in finding equivalent fractions. In the course of solving the problems, students develop strategies for adding and subtracting fractions and mixed numbers. Through class discussion these strategies are made more explicit and efficient. The inverse relationship between addition and subtraction is developed through the exploration of fact families. Although many students understand that addition and subtraction are related in whole-number contexts, they do not always extend this idea to include fractions. The last problem gives students presorted addition and subtraction problems to solve, characterize, and from which to create a general algorithm.

Investigation 3

Multiplying With Fractions

Investigation 3 focuses on developing computational skill with and understanding of fraction multiplication. Various contexts and models are introduced to help students make sense of when multiplication is appropriate. The first two problems develop multiplication with simple fractions and the third and fourth problems focus on fraction, mixed number, and whole number combinations. Estimation is developed across the problems in the investigation, as well as the idea that multiplication does not always lead to a larger product. The last problem is structured like the one used to develop addition and subtraction algorithms. Students are given a set of presorted multiplication problems. Through the process of solving and characterizing how the problems are categorized, students develop a general algorithm for fraction multiplication.

Investigation 4

Dividing With Fractions

This investigation has the same goals as Investigations 2 and 3, except the operation of division is explored. Everyday situations are used to help students make sense of when division is an appropriate operation. The inverse relationship of multiplication and division is explored. The last problem uses presorted division problems to develop a general algorithm for fraction division.

Mathematics Background

Writing Number Sentences

Helping students learn to use mathematical language (i.e., symbols) correctly and with confidence is a goal of the CMP materials. We do this by using symbols connected to contexts so that the context gives the symbols meaning. Using symbolic notation, as well as reformulating symbolic expressions using the rules and syntax of mathematics, can give new insights into problem situations. These are among the fundamental activities of mathematics. Learning the symbolic language of mathematics requires experience to make sense of what the symbols mean and how to operate with them. In CMP, we develop symbolic proficiency over time. You will see students frequently asked to write a number sentence to capture their work. The sentences become more symbolic in grade seven and reach higher levels of sophistication in grade eight units.

Developing Algorithms

Rational numbers are the heart of the middle-grade experiences with number concepts. The concepts of fractions and operations on fractions can be difficult for students. Part of the reason for students' confusion about rational numbers can be a rush to symbol manipulation with fraction operations without time spent in making sense of the concepts and building experiences that show reasons for why the algorithms work. In addition, students need to understand the operations in ways that help them to judge what operation or combination of operations is needed in a given situation. This unit is designed to provide experiences in building algorithms for the four basic operations with fractions, as well as opportunities for students to consider when such operations are useful in solving problems. Building this kind of thinking and reasoning supports the development of skill with the algorithms. By the end of this unit, we expect students to have efficient algorithms for all four basic operations with fractions, including mixed numbers.

The development of algorithms in this unit draws upon concepts and procedures developed in *Bits and Pieces I*. In *Bits and Pieces I*, students developed an understanding of basic interpretations, models, equivalence, and ordering of rational numbers. In *Bits and Pieces II,* we draw upon this development and the models that were introduced—fraction strips or bars, number lines, grid-area, and partition models—because they connect directly to operations on rational numbers. See *Mathematics Background* in the Teacher's Guide of *Bits and Pieces I* for a full discussion of the concepts and models introduced.

For all four operations we use the same type of development. The development of algorithms for each operation and the understanding of those algorithms involve experiences with contextual problems, models, strategies, and estimation. Problems in context help students make sense of an operation and how the operation can be computed. The problem contexts lead students to model situations and to write number sentences that are representative of the particular situation so they begin to make sense of when an operation is appropriate. Students are often asked to make estimates and use them to decide if their models and symbolic work are reasonable. By analyzing the diagrams and models they develop and their resulting quantities, and relating this to their symbolic work and their estimates, students begin to develop algorithms for fraction operations. An underlying goal of all this work is learning to both write and read mathematical language.

For each operation, the last problem in the investigation asks students to analyze a set of sorted computation problems that seem to belong together. They compute the problems and then look for strategies that can be used in a case of that sort. These are refined into algorithms that are efficient for all cases. Usually the students end up with algorithms that are like the standard algorithms taught by direct instruction in many programs. But, they also end up with understanding and insight into the operations and when they are useful. Also, students will have useful strategies for computations with particular number situations. For example, students may come to realize that in a multiplication problem if one factor is $\frac{1}{2}$, you can compute the product by doubling the denominator of the other factor because the piece size needs to be half as big. $\frac{3}{8} \times \frac{1}{2} = \frac{3}{16}$.

Estimation

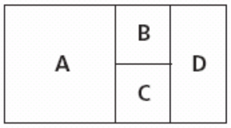
Rather than rush to compute in a given situation, students have experiences with estimating sums, differences, products, and quotients. The initial questions CMP helps students to ask are, "About how big will the answer be? What answer makes sense?" These give students a way to know if their computations are wrong, whether the calculation has been done by hand or by calculator. Developing such a sense of fractions and operations takes a long time. At this point in the curriculum students have had quite a bit of practice finding equivalent fractions and decimals and changing among fractions. They have developed some benchmark fractions that they can use to estimate relative size. Students will use these skills to develop strategies to estimate fraction computations.

The strategies used to estimate can differ. For example, in a situation where the goal is to decide what whole number a sum is closest to or what is a reasonable sum, a useful strategy is to round the numbers to the nearest benchmark. In contrast, you may want to use an estimation strategy that leads to an estimate that you know is too large (overestimate) or too small (underestimate). Consider the situation where you go shopping and you cannot spend more than $20. When you estimate the total cost of the items you want to purchase, you need to estimate in such a way that you can be sure your actual sum is less than $20. Using a purposeful estimation approach, in this case an overestimate, you can know whether or not your actual sum will be less than $20.

Addition and Subtraction

Strategies for operations with fractions can be developed with contexts that help students learn how to put fractions together and take them apart. As students model and symbolize aspects of contextual situations, students develop meaning for and skill in using the operations of addition and subtraction. Additionally, students learn the value of equivalence when changing the representation of fractions to a form with common denominators so that the numerators can be added or subtracted.

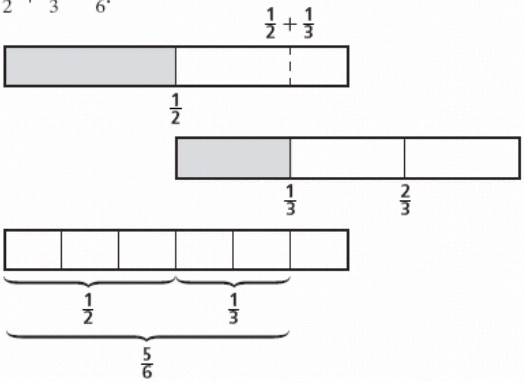
Students' previous work with equivalence and partitioning is critical to the development of strategies for adding or subtracting. The following area model provides a context where both naming fractional parts of a whole and equivalence can emerge as students try to write number sentences to model combining section A with section B.



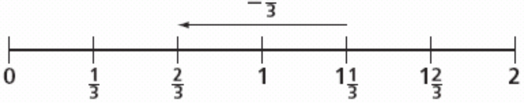
In order to find the sum of A + B, or $\frac{1}{2} + \frac{1}{8}$, students need to use equivalent fractions to rename $\frac{1}{2}$ as $\frac{4}{8}$. The area model helps students visualize A, $\frac{1}{2}$, as 4 eighth-sized sections. By asking students to write number sentences, and to explain how the number sentence helped them arrived at the sum $\frac{5}{8}$, students begin to understand why it is necessary to rename fractions when adding and subtracting and the role that equivalence plays in doing so.

In addition to equivalent fractions, students will need to draw on their understanding of equivalent forms. In *Bits and Pieces I* students worked with the relationship between a mixed number and an improper fraction. As students develop strategies to add and subtract fractions in situations that lead to borrowing or carrying, students learn the value of being able to rewrite fractions in equivalent forms. For example, understanding why $8\frac{2}{3}$ is equivalent to $7\frac{5}{3}$ is critical to understanding how to borrow in fraction situations.

There are other models that can be used to highlight the role of equivalence and support understanding of addition and subtraction. The *fraction-strip model* was used in conjunction with the *number-line model* in *Bits and Pieces I* to develop meaning for fractions and equivalence. Here fraction strips are used to represent $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$.



The *number-line model* helps make the connection to fractions as numbers or quantities. This is a number line for 0 to 2 marked to illustrate $1\frac{1}{3} - \frac{2}{3} = \frac{2}{3}$.

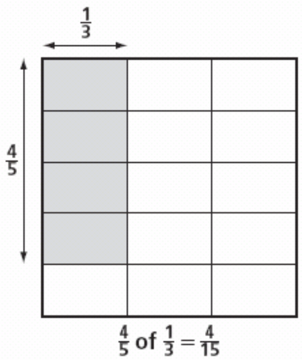


Multiplication

One of the first hurdles for students in their understanding of multiplication of fractions is realizing that multiplication does not always "make larger," as their experience with whole number multiplication has firmly established. In fact, with multiplication of a fraction by a whole number, the fraction can be interpreted as *an operator* that may "stretch" (make larger) or "shrink" (make smaller) depending on whether the fraction is greater or less than 1. This is a big idea that supports understanding what multiplication of fractions entails.

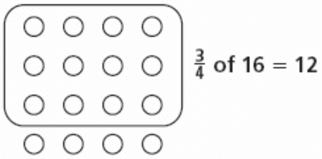
A second hurdle for students is understanding that when they encounter a situation where one needs to take a fraction *of* a quantity, *of* means multiplication. For example, to find $\frac{2}{3}$ of 9, you multiply $\frac{2}{3} \times 9$ to get 6. The temptation is very great to start by telling students this rather than have them encounter the dilemma of the meaning of *of*. In resolving what *of* means in this context, student learning is enhanced.

Models for multiplication of fractions used in the unit are both area models and partitioning. Area models are also useful to help students represent situations, especially multiplication and later decimals and percents. The diagram shows $\frac{4}{5}$ of $\frac{1}{3}$.

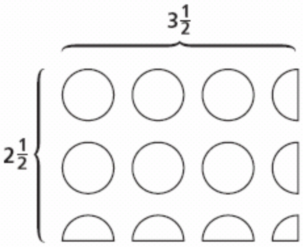


Students also use a model of fraction situations that is based on *partitioning* a number line or strip. The example shows finding $\frac{4}{5}$ of $\frac{1}{3}$ or $\frac{4}{5} \times \frac{1}{3}$. (Figure 1)

You may see students use discrete models to make sense of situations where they are working on discrete objects. An example of a discrete situation is finding $\frac{3}{4}$ of 16 apples. Here each apple represents a separate entity.



Another example, below, shows $2\frac{1}{2} \times 3\frac{1}{2}$.

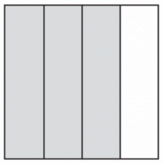


$2\frac{1}{2} \times 3\frac{1}{2} = 8\frac{3}{4}$

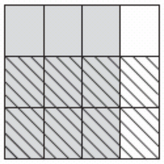
Developing the Multiplication Algorithm

Students notice that multiplication is easy for proper fractions because they can just multiply the numerators and multiply the denominators. However, they have little understanding of why this works. The models we have discussed can help you support understanding. The following looks at both the area and the number-line model as a means of understanding why the algorithm works.

Consider the problem $\frac{2}{3} \times \frac{3}{4}$. To show $\frac{2}{3} \times \frac{3}{4}$ with an area model, first represent the $\frac{3}{4}$ by dividing a square into fourths and shading three of the fourths.

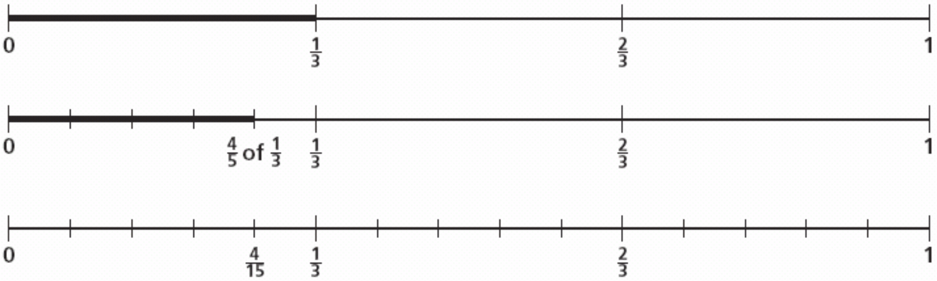


To represent taking $\frac{2}{3}$ of $\frac{3}{4}$, divide the whole into thirds by cutting the square the opposite way, then shade two of the three sections. The part where the shaded sections overlap represents the product, $\frac{6}{12}$.



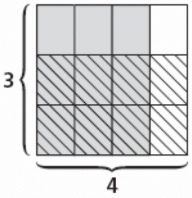
Note what happens to the numerator and the denominator when you partition and how this is related to the algorithm for multiplying fractions. When the square is partitioned, the denominators are used to partition and repartition the whole. In this problem, there are fourths or four parts. When the fourths are partitioned into thirds, or three parts each, the number of pieces in the

Figure 1



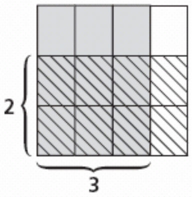
$\frac{4}{5}$ of $\frac{1}{3} = \frac{4}{15}$

whole triples so there are 12 pieces. In the algorithm, when you multiply the denominators (3×4), you are resizing the whole to have the correct number of parts. This means that the denominator in the product has the same role as the denominator in a single fraction. The role is to determine how many parts are in the whole.



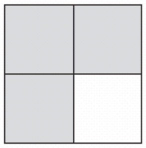
denominator → $\frac{2}{3} \times \frac{3}{4} = \frac{2 \times 3}{3 \times 4} = \frac{6}{12}$

Likewise, the numerator is keeping track of how many of the one-twelfth parts are being referenced. During the original partitioning, $\frac{3}{4}$, or 3 fourth-sized parts, were shaded. In order to take $\frac{2}{3}$ of the 3 one-fourth sized parts, you have to take 2 of the one-twelfth sections from each of the 3 one-fourth sized parts. This can be represented by the product of the numerators 2×3.



numerator → $\frac{2}{3} \times \frac{3}{4} = \frac{2 \times 3}{3 \times 4} = \frac{6}{12}$

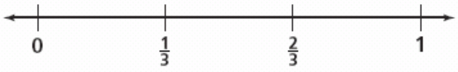
Note that dividing a square with both horizontal and vertical lines for the first fraction does not lead to the kind of partitioning that suggests multiplication of numerators and denominators. For example, if you represent $\frac{3}{4}$ like this:



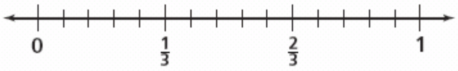
you may find $\frac{2}{3}$ of $\frac{3}{4}$ by noticing that there are three pieces shaded and you are concerned with 2 of them, so the answer is $\frac{2}{4}$. This is a perfectly reasonable strategy for this problem. The question is whether this strategy will *always* work no matter what fractions. For $\frac{1}{5} \times \frac{2}{3}$ this is not a helpful strategy.

The following illustrates how the *number-line model* is helpful for $\frac{1}{5} \times \frac{2}{3} = \frac{2}{15}$ and is generalizable, even if tedious with large numerators or denominators.

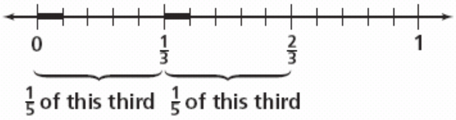
Draw a number line and label 0 and 1. Partition the number line into thirds and mark $\frac{1}{3}$ and $\frac{2}{3}$.



Now break each third into 5 equal parts to get a total of 15 equal parts.



Each fifth of a third is $\frac{1}{15}$, so the two parts marked would be $\frac{2}{15}$. Again the product of the numerators gives the numerator of the product and the product of the denominators gives the denominator of the product.



Using Distribution as a Strategy to Multiply Fractions

Another approach to multiplication of fractions that students use is based on the distributive property. This approach is often used when mixed numbers are involved. Many students use the ideas of breaking a number apart quite intuitively, but often do so incorrectly. The terminology of distribution is not important for students to know at this time. However, because students often use this strategy incorrectly, we provide an opportunity to talk about it in Problem 3.4. We do not wish to promote it as the only approach to multiplying fractions, but as one that is sensible in some situations.

Here is an algorithm for whole number multiplication that uses this approach. Consider the problem 32 × 24.

$$\begin{array}{rl} 32 & \\ \times\ \underline{24} & \\ 8 & \leftarrow 4 \times 2 \\ 120 & \leftarrow 4 \times 30 \\ 40 & \leftarrow 20 \times 2 \\ \underline{600} & \leftarrow 20 \times 30 \\ 768 & \end{array}$$

This approach is very much like multiplying binomials in algebra. It involves breaking up both numbers into their respective tens and ones. With 32 × 24 it looks like this:

$$\begin{tikzcd}[column sep=tiny] & & 30 \times 20 = 600 & \\ (30 \arrow[rr, bend left] \arrow[rrr, bend right] & + 2) \times & (20 & + 4) \\ & & 30 \times 4 = 120 & \end{tikzcd} \qquad \begin{tikzcd}[column sep=tiny] & & 2 \times 20 = 40 & \\ (30 + & 2) \times \arrow[r, bend left] \arrow[rr, bend right] & (20 & + 4) \\ & & 2 \times 4 = 8 & \end{tikzcd}$$

First, multiply 30 times 20 (tens place in 24) and then 30 times 4 (ones place in 24). Next, multiply by the 2 in the ones place of 32. Multiply 2 times 20 (tens place in 24) followed by 2 times 4 (ones place in 24). Finally, total each partial product (600 + 120 + 40 + 8) to arrive at a total product of 768.

With a problem like $2\frac{1}{2} \times 2\frac{1}{4}$, students may break up each factor and try to work with $(2 + \frac{1}{2}) \times (2 + \frac{1}{4})$. If they distribute correctly, they would reason as shown next.

$$\begin{tikzcd}[column sep=tiny] & & 2 \times 2 = 4 & \\ (2 \arrow[rr, bend left] \arrow[rrr, bend right] & + \frac{1}{2}) \times & (2 & + \frac{1}{4}) \\ & & 2 \times \frac{1}{4} = \frac{1}{2} & \end{tikzcd}$$

and

$$\begin{tikzcd}[column sep=tiny] & & \frac{1}{2} \times 2 = 1 & & \\ (2 + & \frac{1}{2}) \times \arrow[r, bend left] \arrow[rr, bend right] & (2 & + \frac{1}{4}) \arrow[r] & 4 + \frac{1}{2} + 1 + \frac{1}{8} = 5\frac{5}{8} \\ & & \frac{1}{2} \times \frac{1}{4} = \frac{1}{8} & & \end{tikzcd}$$

Another approach that makes sense with this problem is to work with $(2 + \frac{1}{2}) \times 2\frac{1}{4}$. If you only break up the first factor the reasoning is as follows:

$$\begin{tikzcd}[column sep=tiny] & 2 \times 2\frac{1}{4} = 4\frac{1}{2} & & \\ (2 \arrow[rr, bend left] & + \frac{1}{2}) \times \arrow[r, bend right] & 2\frac{1}{4} \arrow[r] & 4\frac{1}{2} + 1\frac{1}{8} = 5\frac{5}{8} \\ & \frac{1}{2} \times 2\frac{1}{4} = 1\frac{1}{8} & & \end{tikzcd}$$

Division

Division also has its share of conceptual difficulties. The answer to a division involving fractions is not necessarily smaller than the dividend. Again it depends on the size of the fraction for both the dividend and the divisor. For example, $3 \div \frac{1}{3} = 9$ and $\frac{1}{4} \div \frac{1}{3} = \frac{3}{4}$ result in a quotient that is larger than the dividend or the divisor. However, in $\frac{1}{3} \div 9 = \frac{1}{27}$, the quotient is smaller than either the dividend or the divisor, and in $\frac{1}{4} \div \frac{3}{4} = \frac{1}{3}$, the quotient is between the dividend and the divisor! Examination of division of fractions in context can help students build an understanding of the operation as well as skill in predicting (or estimating) the kind of answer expected.

In the development of meaning of operations, we ask students to write problems that fit a given computation expression. This will tell you a lot about whether students can interpret different kinds of division situations and whether they can make sense of what the answer to a division problem, including its fractional part, means in a given situation.

In order for students to make sense of any division algorithm, they need to think about what the problem is asking. Creating diagrams to model division problems is a key part of developing this understanding. There are cases where the use of pictorial reasoning is more efficient or just as efficient as an algorithm. Also, the development of an efficient algorithm is tied to one's ability to understand pictorially and linguistically what the problem is asking. As students work toward trying to develop and use algorithms they may continue to draw pictures to help them think through the problem. However, they also need to learn to talk about what the problem is asking, what the answer means, what makes sense as a solution strategy, and how this language is related to the algorithm.

Our goal is to help students develop an efficient algorithm. Not all students may get to the "reciprocal" algorithm for division of fractions, but they should have efficient strategies that make sense to them to solve problems that call for division with fractions.

Understanding Division as an Operation

There are two situations associated with division. We can focus on division as a *sharing* operation in problems like this:

> Ms. Li brings peanuts to be shared equally by members of groups winning each game. How much of a pound of peanuts will each student get when the peanuts weigh $\frac{1}{2}$ pound and four students are on the winning team?

Here the question is how much each of the four team members will get if the amount is shared equally. You can also think of this as a *partitioning,* sometimes called *partitive,* model.

Another kind of situation calling for division is a *grouping* situation. For example:

> Naylah plans to make small cheese pizzas to sell at a school fundraiser. She has nine bars of cheese. How many pizzas can she make if each takes $\frac{1}{3}$ bar of cheese?

Here the question is how many groups of size $\frac{1}{3}$ can be made from nine bars of cheese? Another way to ask this is "How many $\frac{1}{3}$'s are in 9?" This kind of problem has multiple names—*measurement, subtractive,* or *quotitive* model. Knowing these names is not important for your students, but it is important for them to experience situations representing these different interpretations of division. Otherwise students will not have all the tools for deciding *when* division is the appropriate operation.

Developing a Division Algorithm

We develop understanding of division of fractions by looking at three cases—division of a whole number by a fraction, division of a fraction by a whole number, and division of a fraction by a fraction. From these situations, several approaches to division are developed: multiplying by the denominator and dividing by the numerator, multiplying by the reciprocal, and the common denominator approach.

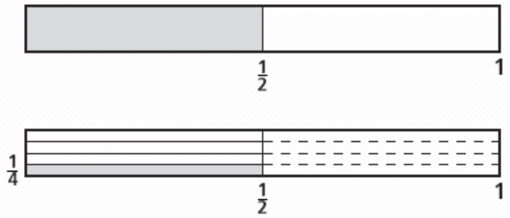
Multiplying by the Denominator and Dividing by the Numerator When you confront a whole number divided by a fraction, such as $9 \div \frac{1}{3}$, it is easiest to interpret this as finding how many $\frac{1}{3}$'s are in 9. To answer, students find how many $\frac{1}{3}$'s are in a whole and multiply by 9 to find the total number of $\frac{1}{3}$'s in 9. The reasoning is as follows: In $9 \div \frac{1}{3}$, I have to find the total number of $\frac{1}{3}$'s in 9. I know that there are three $\frac{1}{3}$'s in 1, so there are 9×3 or 27 of the $\frac{1}{3}$'s in 9. In summary, $9 \div \frac{1}{3} = 9 \times 3 = 27$.

Next we move to $9 \div \frac{2}{3}$. A key to understanding in the development of division of fractions is the relationship between the two problems $9 \div \frac{1}{3}$ and $9 \div \frac{2}{3}$. The question is, how are the answers related and why? The answer to the first problem is 27 and the answer to the second is $13\frac{1}{2}$. Why does it make sense for the answer to the second to be half that of the first? You can interpret the first problem as how many $\frac{1}{3}$'s are in 9 and the second as how many $\frac{2}{3}$'s are in 9. Now it makes sense that it will take twice as much to make $\frac{2}{3}$ than to make $\frac{1}{3}$ so the number you can make will be half as large.

This conversation allows students to begin to relate a whole string of division problems, such as $9 \div \frac{1}{3}$, $9 \div \frac{2}{3}$, $9 \div \frac{3}{3}$, and $9 \div \frac{4}{3}$. Here we are building a case for thinking of division of fractions as multiplying by the denominator of the divisor to find how many in one whole and then dividing by the numerator because that is how many it takes to make a piece of the required size. These two actions are the same as multiplying by the reciprocal.

When moving to other cases, such as dividing a fraction by a whole number and dividing a fraction by a fraction, support student thinking with models. In these situations, we continue to see that multiplying by the denominator and dividing by the numerator makes sense because we can interpret what each part is accomplishing. In the computation $\frac{2}{3} \div \frac{3}{4}$, we can find the "answer" by multiplying by $\frac{4}{3}$. But what does this mean? Multiplying by 4 tells us how many $\frac{1}{4}$'s are in a whole and dividing by 3 adjusts this answer by accounting for the fact that it takes 3 of the $\frac{1}{4}$'s to make one object of the size the problem requires. We have found that many students are able to see the pattern of "multiply by the denominator and divide by the numerator of the divisor" and explain why it makes sense through this kind of classroom talk.

Multiplying by the Reciprocal The reciprocal approach may arise when working with fraction divided by whole number contexts. For example, with the problem $\frac{1}{2} \div 4$, students often draw the following diagram.



They may reason by saying, "I divided the $\frac{1}{2}$ into four parts so I could find $\frac{1}{4}$ of the half." Here students are relating the problem $\frac{1}{2} \div 4$ to the problem $\frac{1}{2} \times \frac{1}{4}$. This type of reasoning, the diagram that develops it, and the number sentences that support it, help students move from the division problem to multiplying by the reciprocal.

Common Denominator Approach Some students intuitively try the same approach for division that worked in addition and subtraction—finding a common denominator. This algorithm nicely links to their whole number understanding of division. For example, in the problem $\frac{7}{9} \div \frac{1}{3}$, students rename the fractions to say $\frac{7}{9} \div \frac{3}{9}$. The common denominator allows them to reason that if they have 7 one-ninth sized pieces of something and want to find out how many groups of 3 one-ninth sized pieces they can make, they can find the answer from the computation $7 \div 3$, which equals $2\frac{1}{3}$. This algorithm is used in *Bits and Pieces III* to develop decimal division.

Relating Multiplication and Division

In additive situations, those involving addition and subtraction, the quantities are easy to count, measure, combine and separate. This is because each quantity in an addition or subtraction problem has the same kind of label or is the same type of unit. For example, $3 + 4 = 7$ can be thought of as 3 marbles plus 4 marbles equals 7 marbles. Each quantity is a number of marbles.

In multiplicative situations, those involving multiplication and division, the quantities are not so straightforward. Each number may represent a different kind of unit. For example, if tomatoes cost $0.87 per can, the total cost for 6 cans can be found by multiplying 6 cans × 87 cents. It is hard to imagine a situation where adding tomatoes and money would make sense.

Another challenge is the different kinds of situations that call for multiplication and for division. A multiplication problem may be counting an array, or finding an area, or finding the sum of a repeated addition, and so on. Division may be finding how many groups of a certain size or measure that you can make from a given quantity or how many objects or parts would be in each of a given number of groups.

For example, the number sentence $3 \times 4 = 12$ could represent 3 people, each with 4 candies. The same number sentence could also represent 3 candies given to each of 4 people. The two types of division situations, sharing and grouping, are related to these two multiplication situations. The diagram in Figure 2 shows the grouping model of division first followed by the sharing model.

It is important that students develop a sense of the kinds of situations for which each operation is useful. Therefore, you will see attention to meanings and interpretation of the operation in the unit.

Inverse Relationships

Fact families and missing-value problems are used to introduce the inverse relationships of addition and subtraction, and multiplication and division. In elementary grades, students learn about *fact families* for whole numbers. For example, they learn that the addition problems $3 + 5$ and $5 + 3$ both equal 8. In addition, they are related to two subtraction sentences, $8 - 5 = 3$ and $8 - 3 = 5$. In this unit these ideas are expanded to include fractions. For example, students learn that the addition sentence $\frac{7}{10} + \frac{1}{2} = \frac{6}{5}$ is related to $\frac{1}{2} + \frac{7}{10} = \frac{6}{5}$ and two subtraction sentences, $\frac{6}{5} - \frac{1}{2} = \frac{7}{10}$ and $\frac{6}{5} - \frac{7}{10} = \frac{1}{2}$.

Understanding the inverse relationship between the operations pairs of addition-subtraction and multiplication-division is a tool that lends itself to many situations, one of which is equation solving. In this unit, missing-value problems are used to introduce students to the use of a variable as a placeholder. However, the focus is on understanding inverse relationships. We do not expect students to develop formal procedures or notation for solving algebraic equations at this stage.

Missing-value problems, as used in this unit, will help students begin to develop a generalized understanding of inverse relationships. This generalization is aided by working on these relationships in non-whole number contexts. In whole number contexts, such as $20 \div N = 5$, solving for N is partly aided by students using multiplication and division facts with which they had repeated experience. In a problem like $\frac{8}{15} \div N = \frac{2}{3}$, this becomes more difficult. Students have to think about which values are the factors in the related multiplication problems $\frac{2}{3} \times N = \frac{8}{15}$ and $N \times \frac{2}{3} = \frac{8}{15}$. Here, N and $\frac{2}{3}$ are the factors and $\frac{8}{15}$ is the product of the related multiplication problem. Going a step further, if you divide the term that represents the product of the multiplication problem by one of the factors you will get the other factor. This leads to the related missing-value problem $\frac{8}{15} \div \frac{2}{3} = N$ or the realization that if you divide $\frac{8}{15}$ by the known value $\frac{2}{3}$, you will get the other factor, N.

Keep in mind that inverse relationships will be explored in later units with other number contexts such as decimals and integers. For most students this will be an initial introduction and mastery is not expected. But over time, students will start to think beyond the actual numbers to the relationships that exist among the values in related addition and subtraction problems and multiplication and division problems, or fact families. This understanding will be a powerful tool for students to use in other mathematical contexts.

Figure 2

Grouping

3	×	4	=	12	→	12	÷	4	=	3
Unit		*Rate*		*Total*		*Total*		*Rate*		*Unit*
Number of people		Candy per person		Candies		Candies		Candy per person		Number of people

Sharing

3	×	4	=	12	→	12	÷	4	=	3
Rate		*Unit*		*Total*		*Total*		*Rate*		*Unit*
Candy per person		Number of people		Candies		Candies		Number of people		Candy per person