Unit Introduction

Accentuate the Negative

Integers and Rational Numbers

Goals of the Unit

- Use appropriate notation to indicate positive and negative numbers
- Locate rational numbers (positive and negative fractions and decimals and zero) on a number line
- Compare and order rational numbers
- Understand the relationship between a positive or negative number and its opposite (additive inverse)
- Develop algorithms for adding, subtracting, multiplying, and dividing positive and negative numbers
- Write mathematics sentences to show relationships
- Write and use related fact families for addition/subtraction and multiplication/division to solve simple equations with missing facts
- Use parentheses and order of operations to make computational sequences clear
- Understand and use the Commutative Property for addition and multiplication of positive and negative numbers
- Apply the Distributive Property with positive and negative numbers to simplify expressions and solve problems
- Use positive and negative numbers to graph in four quadrants and to model and answer questions about applied settings

Developing Students' Mathematical Habits

The overall goal of *Connected Mathematics* is to help students develop sound mathematical habits. Through their work in this number unit, students learn important questions to ask themselves about any situation that can be represented and modeled mathematically, such as:

- *How do negative and positive numbers help in describing the situation?*
- *What will addition, subtraction, multiplication, or division of positive and negative numbers tell about the problem?*
- *What model(s) for positive and negative numbers would help in showing the relationships in the problem situation?*

Mathematics of the Unit

Pearson Prentice Hall Professional Development

Overview

In the middle grades, students are introduced to fractions and decimals. The next major hurdle is building an understanding of positive and negative numbers, i.e., integers, fractions, and decimals. Students have experienced these kinds of numbers informally in their everyday world. For example, temperatures drop below zero in the winter or soar above 90 degrees in the summer, and sports teams are said to be ahead or behind by so much. Students have intuitively used operations on integers to make sense of these situations. This unit explores situations that require representation with positive and negative numbers. These situations motivate more formal ways to add, subtract, multiply, and divide these numbers. Students formalize algorithms for operating using positive and negative numbers. They also consider the order of operations and selected properties.

Summary of Investigations

Investigation 1

Extending the Number System

This investigation gives students experiences with positive and negative numbers, ordering, and informal operations in a variety of contexts so that subsequent formal work can be based on "what makes sense." Positive and negative numbers in the form of integers, fractions, and decimals are also represented on a number line.

Investigation 2

Adding and Subtracting Integers

Students build on the informal work of Investigation 1 to formulate algorithms for addition and subtraction of positive and negative numbers. In each problem, students are encouraged to think about the meaning of the operations from several perspectives and use different representation models (number line and chip board).

Investigation 3

Multiplying and Dividing Integers

This investigation is structured and developed in a style parallel to that of Investigation 2. The number line model and fact families as well as the contexts of time, distance, and speed are used to develop students' understanding of multiplication and division of positive and negative numbers. Since students have not done much informal multiplication and division, a useful context for questions leading to multiplication and division is explained before asking students to formulate algorithms. The context involves motion at various rates in both directions on a number line. The cases of a negative number times a positive number and a negative number divided by a positive number could come quite easily from the chip board context. However, that model doesn't seem to lead naturally to those cases that involve the product or quotient of two negatives. In almost any context, you have to think hard to get a reasonable guide to the operations we want to develop. In the time, rate, distance, and position setting, these ideas are plausible. The cases of combinations of "signs" for multiplication are explained by looking at number patterns.

Investigation 4

Properties of Operations

Students compare algebraic properties of the operations on positive and negative numbers (i.e. the rational numbers) to those of the number system of only positive numbers (whole numbers). It's not intended to be a full-scale treatment of field properties of the real numbers.

Mathematics Background

Most students may be able to add, subtract, multiply, and divide whole numbers, fractions, and decimals. However, most have not been asked to think about what the operations mean and what kinds of situations call for which operation. Students need the development of the disposition to seek ways of making sense of mathematical ideas and skills. Otherwise, they may end up with technical skills but without ways of deciding when and how those skills can be used to solve problems.

One way to develop the desire to make sense of these ideas is to model such thinking in classroom conversation. Asking questions about meaning, (about what makes sense) as a regular, expected part of classroom discourse helps focus students on making connections. Exploring new aspects of numbers in a way that builds on and connects to what they already know is likely to have two good effects. First, students will deepen their understanding of familiar numbers and operations. Second, the new numbers, integers, and negative rational numbers will be more deeply integrated into students' own mathematical knowledge and resources.

Students find several things difficult about working with positive and negative numbers.

- The fact that -27 is less than -12 is contrary to students' experience with whole numbers (positive integers and zero). This understanding requires building mental images and models that allow students to visualize these new comparisons and relationships.
- The operation of subtraction, especially of subtracting a negative number, is difficult for students to understand. In this unit, students will have several opportunities to think about what makes sense and why. They will encounter representations and models that will help them better understand subtraction.
- The idea that subtracting a negative number gives the same result as adding the opposite of the negative number (adding a positive) is difficult for many students. This understanding must develop over time as students make observations and comparisons between subtraction and addition. Recognizing that these are inverse operations and that addition sentences are related to subtraction sentences helps students to expand their understanding of this concept.
- Multiplying two negatives and getting a positive number does not make sense to many students. In fact, the usual ways of giving meaning to multiplication, such as repeatedly adding an amount, seem of little help in making sense of $-12 \times (-5)$.
- In the unit, we approach these difficult concepts of subtracting a negative and multiplying two negatives through the use of set and number line models and the relationships that exist between addition and subtraction and multiplication and division.
- A number of other confusions occur. For example, the idea that a negative rational number names a point on the number line or knowing that $-3\frac{1}{2}$ can be thought of as $-3 + (-\frac{1}{2})$ is often not transparent.

Using Models for Integers and the Operations of Addition and Subtraction

The number line is a model that is used throughout the number units. It was first introduced in *Bits and Pieces I* to develop understanding of equivalence of fractions and decimals. It was used in *Bits and Pieces II and III* to help develop the operations for fractions and decimals. It is used later in *Looking for Pythagoras* to introduce square roots and irrational numbers. In this unit, students use the directed distance model with the number line to visualize adding and subtracting integers. Here are two situations that students encounter that use both *distance* and *direction* as ways to consider integers.

> *The world record for fastest rise in outside air temperature occurred in Spearfish, South Dakota, on January 22, 1943. The temperature rose from $^{-}4$°F to $^{+}45$°F in two minutes. What was the change in temperature over those two minutes?*

On a number line, this change can be shown using an arrow. (Figure 1)

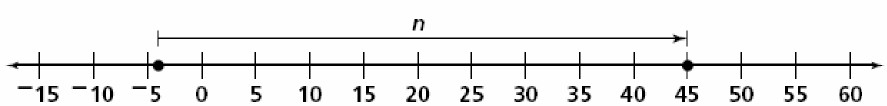
A student might reason: "From $^{-}4$°F to 0°F is an increase of $^{+}4$°F, and from 0°F to 45°F is an increase of $^{+}45$°F. So the total change is an increase of $^{+}49$°F." The situation can be described with a number sentence:

$$^{-}4^{\circ} + n = {}^{+}45^{\circ} \quad \text{or} \quad {}^{-}4^{\circ} + {}^{+}49^{\circ} = {}^{+}45^{\circ}$$

The sign of the change in temperature shows the direction of the change. If the temperature had dropped 10°F, the student would write the change as $^{-}10$°F to show the size and direction of the change. (Figure 2)

$$^{-}4^{\circ} + n = {}^{-}14^{\circ} \quad \text{or} \quad {}^{-}4^{\circ} + {}^{-}10^{\circ} = {}^{-}14^{\circ}$$

Figure 1



We can write these equations without the degree markers. We just have to remember what the answer means. To facilitate the development of the algorithms, the absolute value concept is introduced in Investigation 2 as a way to talk about distance on the number line. It also helps to talk about the value of a number when direction is not considered.

Colored chips can also be used to develop a strategy for adding and subtracting integers. Using this model requires an understanding of opposites. For example, $^-3$ and $^+3$ are opposite because $^+3 + {}^-3 = 0$, or each number is equidistant from the origin on the number line. Red-black pairs represent opposites ($^-1$ and $^+1$), which add to 0. The chip model uses one color of chips (black) to represent positive integers and another color (red) to represent negative integers. (**Note:** You may use any collection of two-color chips—designate which color is positive and which color is negative.)

To use the model with addition, begin with an empty chip board. Place chips on the board to represent each addend. If the integer is positive, place that number of black chips on the board. If the integer is negative, place that number of red chips on the board. If the two integers being added have the same sign, the sum is the total number of chips on the board. For example, to add $^-4 + {}^-3$, place 4 red chips and then another 3 red chips on the board for a total of 7 red chips (representing a sum of $^-7$, or $^-4 + {}^-3 = {}^-7$).

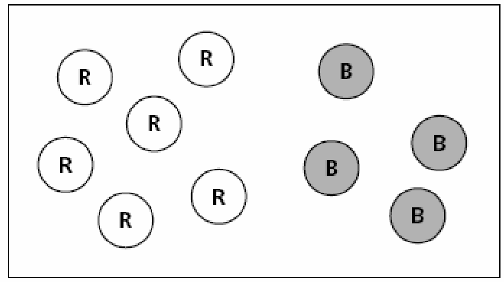
If the integers being added have different signs, place the appropriate number of red and black chips on the board to represent each addend. Simplify the board by removing red-black (opposite) pairs of chips. The chips that remain unmatched represent the sum of the two integers.

Consider this problem:

> *Linda owes her sister \$6 for helping her cut the lawn. She earns \$4 delivering papers with her brother. Is she "in the red" or "in the black"?*

Using collections of black and red chips on a chip board, you can represent the combination of expense and income.

Chip Board



The result, or net worth, is that Linda is "in the red" 2 dollars, or $^-2$ dollars. This problem may be represented with the number sentence, $^-6 + {}^+4 = {}^-2$.

Because each chip represents 1 unit, either positive or negative, red and black chips are thought of as opposites. Combining two opposite chips makes zero ($^+1 + {}^-1 = 0$). In this problem, we can rewrite $^-6$ as $^-2 + {}^-4$ so that 4 chips of each color can be paired to make zeros ($^-6 + {}^+4 = {}^-2 + {}^-4 + {}^+4 = {}^-2 + 0$). After the paired chips are removed, 2 red chips remain. These chips represent the sum $^-2$ ($^-2 + 0 = {}^-2$).

Here is another problem that can be modeled and solved using chips:

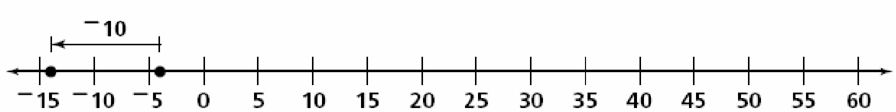
> *Jeremy earns \$10 mowing a lawn. He needs to pay \$15 to rent his equipment. How much more money does he need to pay the rental cost?*

This problem may be modeled using chips by representing the \$10 with a combination of 15 black and 5 red chips ($10 = {}^-5 + 15$). Now \$15 or 15 black chips can be "taken away" [$^-5 + (15 - 15)$], leaving 5 red chips to represent the $^-5$ that Jeremy is "short." Two different number sentences are applicable:

$^+10 + {}^-15 = {}^-5$ and $^+10 - {}^+15 = {}^-5$

As students work on adding and subtracting integers, they notice that it may be helpful to restate an addition problem as a subtraction

Figure 2



problem or vice versa. This involves using opposites of numbers. For example:

To calculate $^{+}12 + {}^{-}8$, they may realize that the result is the same as when they subtract $^{+}8$ in the problem $^{+}12 - {}^{+}8$.

For $^{+}5 - {}^{-}7$, they may realize that the result is the same as when they add $^{+}7$ in $^{+}5 + {}^{+}7$.

Students build on generalizations made using models of integers to extend their work with negative and positive rational numbers.

Fact Families

Fact families are used in this unit to help students understand the relationship between addition and subtraction and between multiplication and division. Fact families were introduced in the number units in grade 6. Here is an example of a fact family:

$^{-}7 + {}^{+}2 = {}^{-}5 \qquad {}^{-}5 - {}^{+}2 = {}^{-}7$

$^{+}2 + {}^{-}7 = {}^{-}5 \qquad {}^{-}5 - {}^{-}7 = {}^{+}2$

Fact families are also used to find a missing factor or addend such as:

$^{+}4 + n = {}^{+}43$ or $^{-}6n = 42$

Models and the Operations of Multiplication and Division

Multiplication can be explored using a number line model and "counting" occurrences of fixed-size movement along the number line. Direction of movement introduces negative and positive movements. For example:

Hahn passes the 0 point running 5 meters per second to the right. Where is he 10 seconds later?

Aurelia passes the 0 point running to the left at 6 meters per second. Where is she 8 seconds later?

Relating division to multiplication helps develop division with integers. A multiplication fact can be used as the basis for creating two related division facts. By developing division this way, students can determine the sign of the answer to a division problem. For example, since we know that $5 \times ({}^{-}2) = {}^{-}10$ (or $^{-}2 \times 5 = {}^{-}10$), we can write related division sentences:

$^{-}10 \div ({}^{-}2) = 5$ and $^{-}10 \div 5 = {}^{-}2$

Then, students can generalize rules for handling the sign of the quotient in a division problem.

Some Notes on Notation

Writing integers with raised signs avoids confusion with symbols for addition and subtraction. However, most software and most writing in mathematics do not use the raised signs.

Positive numbers are usually written without a sign.

$$^{+}3 = 3 \text{ and } {}^{+}7.5 = 7.5.$$

Negative numbers are usually written with a dash like a subtraction sign.

$$^{-}3 = {}^{-}3 \text{ and } {}^{-}7.5 = {}^{-}7.5.$$

Parentheses can help.

$$^{-}5 - {}^{-}8 = {}^{-}5 - {}^{-}8 = {}^{-}5 - ({}^{-}8)$$

The subtraction symbol also indicates the opposite of a number. For example, $^{-}8$ represents the opposite of 8 and $^{-}({}^{-}8)$ the opposite of $^{-}8$.

$$^{-}({}^{-}8) = 8$$

We use raised signs for the first two investigations, after which we use the standard notation.

For multiplication, you can use a raised dot symbol. For example, $3 \times 5 = 3 \cdot 5$. Some students might have seen $3 \cdot (4 + 5)$ or $3 \times (4 + 5)$, or even $3(4 + 5)$.

Order of Operations and Properties

Order of operations rules are introduced.

1. Compute any expressions within parentheses.

   $({}^{-}7 - 2) + 1 = {}^{-}9 + 1 = {}^{-}8$

   $(1 + 2) \times (-4) = 3 \times ({}^{-}4) = {}^{-}12$

2. Compute any exponents.

   $^{-}2 + 3^2 = {}^{-}2 + 9 = 7$

   $6 - ({}^{-}1 + 4)^2 = 6 - (3)^2 = {}^{-}3$

3. Multiply and divide in order, from left to right.

   Example 1

   $1 + 2 \times 4 =$ Multiplication first

   $1 + 8 = 9$

   Example 2

   $200 \div 10 \times 2 =$ Division first

   $20 \times 2 = 40$ Multiplication second

4. Add and subtract in order, from left to right.

   $1 - 2 + 3 \times 4 =$ Multiplication first

   $1 - 2 + 12 =$ Addition and subtraction

   $^{-}1 + 12 = 11$

The Commutative Property of Addition and Multiplication is introduced. Students find that this property does not hold for subtraction or division of integers. The Distributive Property of Multiplication over Addition or Subtraction is also introduced and modeled through finding areas of rectangles. The Associative Property is explored in an ACE exercise. These properties are revisited in several succeeding units, particularly the algebra units.